THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and seventy-second regular meeting of the Society was held in New York City on Saturday, October 31, 1914, extending through the usual morning and afternoon sessions. The attendance included the following thirty-eight members:

Mr. A. A. Bennett, Professor W. J. Berry, Dr. Emily Coddington, Professor F. N. Cole, Dr. Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. G. M. Green, Professor C. C. Grove, Professor H. E. Hawkes, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Professor James Maclay, Professor Helen A. Merrill, Dr. R. L. Moore, Professor Richard Morris, Mr. G. W. Mullins, Professor W. F. Osgood, Dr. G. A. Pfeiffer, Dr. H. W. Reddick, Miss C. E. Seely, Professor L. P. Siceloff, Professor Clara E. Smith, Professor P. F. Smith, Professor W. B. Stone, Professor H. D. Thompson, Professor Oswald Veblen, Mr. H. E. Webb, Mr. R. A. Wetzel, Professor H. S. White, Miss E. C. Williams, Professor E. B. Wilson.

Vice-President L. P. Eisenhart occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. H. R. Kingston, University of Manitoba; Dr. Edward Kircher, Massachusetts Institute of Technology; Mr. Colin MacLennan, Havana Railway, Light and Power Company; Mr. E. E. Moots, Walla Walla, Wash.; Mr. C. N. Reynolds, Jr., Harvard University; Dr. Joseph Rosenbaum, New Haven, Conn.; Dr. Joseph Slepian, Cornell University; Dr. Anna H. Tappan, Iowa State College; Dr. Mabel M. Young, Wellesley College. Four applications for membership in the Society were received.

The Council submitted a list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual election. A committee was appointed to audit the accounts of the Treasurer for the current year. Arrangements were made to adjust the insurance on the property of the Society destroyed by fire on October 10, and to restore the office to working conditions.

The following papers were read at this meeting:

(1) Dr. G. M. GREEN: "On completely integrable systems of homogeneous linear partial differential equations."

(2) Dr. G. A. PFEIFFER: "Contributions to the conformal geometry of analytic arcs."

(3) Dr. C. A. FISCHER: "Conditions for a minimum of an n-fold integral."

(4) Mr. E. C. KEMBLE: "Note on the definition of work."

(5) Professor H. S. WHITE: "Census of the triad systems on 15 letters."

(6) Professor EDWARD KASNER: "A law of reciprocity in the calculus of variations."

(7) Professor K. P. WILLIAMS: "Concerning a certain totally discontinuous function."

(8) Professor T. H. GRONWALL: "Some remarks on conformal representation."

The paper of Mr. Kemble was communicated to the Society and read by Professor Osgood. In the absence of the authors, the papers of Professors Williams and Gronwall were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The completely integrable systems of homogeneous linear partial differential equations considered in Dr. Green's paper contain any number m of dependent variables $y_1, y_2, \cdots, y_m$, and any number p of independent variables $u_1, u_2, \cdots, u_p$. Any solution y_k of the system is expressible linearly, with constant coefficients, in terms of a fundamental system of n solutions: $y_k = c_1 y_k^{(1)} + c_2 y_k^{(2)} + \cdots + c_n y_k^{(n)}$; and all derivatives of the dependent variables, of any order, are expressible linearly in terms of n primary derivatives. A certain determinant W, formed from the primary derivatives of a fundamental system of solutions, takes the place in this discussion of the wronskian for a single ordinary homogeneous linear differential equation of the nth order. It is proved that the first derivatives of W with respect to the independent variables are given by the equations

$$W_{u_1} = f_1 W, \qquad W_{u_2} = f_2 W, \qquad \cdots, \qquad W_{u_p} = f_p W,$$

where the f's are functions of the coefficients, and possibly of some of their derivatives, of the given system of differential equations. They satisfy the integrability conditions

$\partial f_s/\partial u_t = \partial f_t/\partial u_s$, so that W may be found by a quadrature from the coefficients of the differential equations.

The chief purpose of the paper is to prove that the system of differential equations may always be reduced, and without any difficulty, to a canonical form, in which the f's are all zero. The coefficients of the canonical form are expressions in the coefficients of the original system of differential equations, and are such that they remain unchanged for any transformation of the dependent variables of the form $y_1 = \lambda_1 \bar{y}_1, \cdots, y_m = \lambda_m \bar{y}_m$, where the λ's are arbitrary functions of the independent variables $u_1, \cdots, u_p$. It is important, in Professor Wilczynski's method for investigations in projective differential geometry, to find the above-mentioned seminvariant functions, and the present paper gives a method, always applicable, for calculating them expeditiously.

2. The greater part of Dr. Pfeiffer's paper is devoted to an investigation of the properties of symmetry, i. e., the pairing of points symmetric with respect to an analytic arc, and the associated functional equations. Theorems are given concerning the existence of an analytic arc such that with respect to it one of two given intersecting analytic arcs is the symmetric image of the other. This is "the bisection problem," so called by Kasner. A theorem concerning the "n-sector of two given intersecting analytic arcs" is also obtained.

Closely related results are also given stating necessary and sufficient conditions for the conformal equivalence of certain pairs of intersecting analytic arcs to a rectilinear angle and showing the existence of a unique absolute conformal invariant of a pair of intersecting analytic arcs such that the magnitude of the angle formed by them is commensurable with π. Kasner has already shown that at least one such invariant exists and proved its uniqueness in a few special cases.

3. The necessary conditions for a minimum of a double integral expressed in parametric form have been discussed by Kobb and others. In this paper Dr. Fischer derives the analogous conditions for a minimum of the n-fold integral

$$\int\int \cdots \int F(x_1, x_2, \cdots, x_{n+1}, p_{11}, p_{12}, \cdots, p_{n+1n}) du_1 du_2 \cdots du_n.$$

The analogue of the Lagrange differential equation, and a

variable boundary condition are derived. The variable boundary condition is reduced to a very simple form in the special case where $x_1, x_2, \cdots, x_n$ are the independent variables. Then the analogues of the Legendre and Jacobi conditions are discussed, and sufficient conditions are derived for a permanent sign for the second variation.

4. In Mr. Kemble's paper a difficulty in the usual definition of the work done by a force applied at a point is indicated, and a new definition is suggested. It is proved that the work done as defined is equal to the increase in the kinetic energy of the body to which the force is applied.

5. Special methods from 1852 to 1913 brought to light 12 distinct triad systems on 15 letters. Miss L. D. Cummings added, in 1914, 12 further systems. No attempt had been made to secure an exhaustive list of possible systems. In the present paper Professor White studies first the kind of substitution that can leave a triad system invariant. All such must have as some power a substitution of one or another of seven typical forms. The problem therefore is next to find all systems that are invariant under each of these seven substitutions. In this way only such systems are omitted as may have no group except identity. By this means twenty new systems are found.

6. In a previous paper, Professor Kasner investigated the geometric character of the transversality relations connected with the minimizing of double and simple integrals in space. The main result of the present paper is that with each double integral $\int\int F(x, y, z, p, q)dxdy$ there is associated a simple integral $\int G(x,y,z,y',z')dx$ such that the corresponding transversalities are inverse to each other. Each of the integrand functions F and G determines the other except for a factor involving only x, y, z. Other theorems relate to the case where F and G have the same form, and the case where the two transversalities are orthogonally related.

7. Professor Williams's paper appeared in full in the December BULLETIN.

8. In the present paper, Professor Gronwall proves the following theorems:

I. When the analytic function $z = f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$ effects the conformal representation of the circle $|x| < 1$ on a simple (that is, simply connected and nowhere overlapping) region in the z-plane, the area of this region not exceeding A, then, for $|x| \leqq r < 1$,

$$|f(x)| \leqq \sqrt{\frac{A}{\pi} \cdot \log \frac{1}{1-r^2}} \quad \text{and} \quad |f'(x)| \leqq \sqrt{\frac{A}{\pi}} \cdot \frac{1}{1-r^2},$$

and these upper boundaries of $|f(x)|$ and $|f'(x)|$ cannot be replaced by any smaller ones. Less accurate limitations have been given by Koebe and Courant.

II. When $z = f(x) = 1/x + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ effects the conformal representation of the circle $|x| < 1$ on a simple region in the z-plane containing the point at infinity, then $|f(x)| < 9/4r$ for $|x| = r < 1$. A less accurate limitation has been given by Fricke.

F. N. COLE,
Secretary.

THE TWENTY-SIXTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-sixth regular meeting of the San Francisco Section of the Society was held at the University of California on October 24, 1914. Twenty-two persons were present, including the following members of the Society:

Professor R. E. Allardice, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Professor C. E. Brooks, Dr. Thomas Buck, Professor L. E. Dickson, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor J. H. McDonald, Professor W. A. Manning, Professor H. Ç. Moreno, Professor C. A. Noble, Professor E. W. Ponzer.

The chairman of the Section, Professor Manning, presided at the opening of the meeting; the chairman-elect, Professor Haskell, then took the chair. The following officers were elected for the ensuing year: chairman, Professor Haskell; secretary, Dr. Buck; programme committee, Professors Manning and Blichfeldt, and Dr. Buck.

It was voted to omit the spring meeting of the Section because of the summer meeting of the Society to be held in San Francisco, and to hold the fall meeting at Stanford University at a date to be determined later.

The question of a colloquium in connection with the summer meeting in 1915 was discussed, and it was decided to petition the Council to arrange for such a colloquium.

The members took luncheon together as usual.

The following papers were presented at this meeting:

(1) Dr. FRANK IRWIN: "Relation between the zeros of a rational integral function and its derivative."

(2) Professor M. W. HASKELL: "The maximum number of stationary points of algebraic curves."

(3) Professor D. N. LEHMER: "Some further results in the theory of the continued fraction representing the surd $R^{1/\lambda}$" (second preliminary report).

(4) Professor L. E. DICKSON: "On formal and modular invariants."

(5) Professor L. E. DICKSON: "On modular curves."

Abstracts of the papers follow below.

1. The proposition that the roots of the derivative of a polynomial lie, in the complex plane, inside the smallest convex polygon containing the roots of the polynomial itself is well known; but Hayashi's proof recently published in the *Annals of Mathematics* is the first, so far as known, not based on dynamical conceptions. Dr. Irwin points out that an extremely simple proof of this kind may be given.

2. Professor Haskell shows that the maximum number of cusps possible for a plane curve of order m is the greatest integer in $m(m-2)/3$; and that in every case there is a self-dual curve with this maximum number of cusps. The cases $m = 4$ and $m = 6$ are exceptions, the maximum number of cusps being 3 and 9 respectively.

In the case of space curves, the Plücker equations can be satisfied by the same maximum number of stationary points, but it can be shown that the curves are then either plane curves or reducible.

3. Representing the complete quotient in the expansion of $R^{1/\lambda}$ in a continued fraction by

$$\frac{P_n + A_n R^{1/\lambda} + B_n R^{2/\lambda} + C_n R^{3/\lambda} + \cdots + K_n R^{(\lambda-2)/\lambda} + L_n R^{(\lambda-1)/\lambda}}{Q_n}$$

and the corresponding convergent by α_n/β_n, Professor Lehmer has obtained certain interesting inequalities connecting the P's and Q's, which show that there can be only a finite number of P's which have the same value. In a former paper it was shown that the Q's satisfy the indeterminate equation

$$(-1)^{n-1} Q_n = \alpha_n^\lambda - R\gamma_n^\lambda,$$

and by a general theorem due to Axel Thue (Christiania, *Videnskabs-Selskabet Skrifter*, 1908, No. 3), there can be only a finite number of Q's having the same value in the expansion. This important result has not yet been derived from the discussion of the continued fraction itself.

4. The first paper by Professor Dickson gave a survey of the main results in the theory of invariants arising in the theory of numbers. Special attention was given to the construction of formal modular invariants from the geometrical standpoint developed in the October number of the *Transactions*.

5. The second paper by Professor Dickson related to the theory of modular cubic and quartic curves for the interesting case in which the modulus is 2. Such a quartic curve has at most seven bitangents (and aside from special cases exactly seven) whose intersections are either singular points or points with indeterminate polars. In general, all such points are intersections of bitangents. The equivalence of two quartic curves can be decided from a knowledge of their real points, their singular points, and their points with indeterminate polars.

THOMAS BUCK,
Secretary of the Section.

MODULAR INVARIANT PROCESSES.

BY PROFESSOR O. E. GLENN.

(Read before the American Mathematical Society, September 8, 1914.)

Introduction.

LET $f = a_0 x_1^n + \cdots$ be an ordinary algebraical quantic in m variables. Suppose that it is subjected to linear trans-

formations whose coefficients are parameters representing positive residues of a prime number p. The result, $f' = a_0'x_1'^n + \cdots$, will be a quantic whose coefficients will be linear forms in the variables $a_0, a_1, \cdots$ with integral coefficients. Any function φ of the coefficients and variables

$$\varphi = \varphi(a_0, a_1, \cdots; x_1, x_2, \cdots)$$

which possesses the property

$$\varphi(a_0', a_1', \cdots; x_1', x_2', \cdots) \equiv \rho^k \varphi(a_0, a_1, \cdots; x_1, x_2, \cdots) \pmod{p}$$

identically in the a's and x's, ρ being the modulus of the transformation, is called a formal modular covariant of f,* or a formal covariant modulo p of f.

It is the purpose of this paper to develop some invariant processes which produce concomitants of this type; that is, processes which are characteristic of the invariant theory of modular transformations.

§ 1. *Modular Polars.*

It is easy to prove that with reference to m-ary transformations with integral coefficients modulo p the following set of functions is cogredient to the set of variables $x_1, x_2, \cdots, x_m$:

$$x_1^{p^t}, x_2^{p^t}, \cdots, x_m^{p^t} \qquad (t \text{ a positive integer}).$$

In fact if the transformations are

$$x_i = \lambda_i x_1' + \mu_i x_2' + \cdots + \sigma_i x_m' \quad (i = 1, \cdots, m), \tag{1}$$

we have by the multinomial theorem

$$x_i^{p^t} \equiv \lambda_i^{p^t} x_1'^{p^t} + \mu_i^{p^t} x_2'^{p^t} + \cdots + \sigma_i^{p^t} x_m'^{p^t} \pmod{p}.$$

Hence by Fermat's theorem

$$x_i^{p^t} \equiv \lambda_i x_1'^{p^t} + \mu_i x_2'^{p^t} + \cdots + \sigma_i x_m'^{p^t} \pmod{p}, \tag{2}$$

which proves the statement.

In any formal-modular invariant function $\varphi(x_i)$ we may replace the variables x_i by $x_i + lx_i^{p^t}$ without disturbing the property of invariance. Hence follows the theorem.

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* Hurwitz, *Archiv der Math. und Phys.*, ser. 3, vol. 5 (1903), p. 17. Dickson, Madison Colloquium Lectures, Lecture III, and Lecture IV, p. 68.

THEOREM 1: *The modular polar*

$$(3)\qquad E_m^{(t)} \equiv x_1^{p^t}\frac{\partial}{\partial x_1} + x_2^{p^t}\frac{\partial}{\partial x_2} + \cdots + x_m^{p^t}\frac{\partial}{\partial x_m} \pmod p$$

is an invariant operator.

Any other set of functions $f_1, f_2, \cdots, f_m$ which possesses the property of cogrediency with the variables will furnish a modular polar operator.

For illustration consider the modular polars of the quadratic form in m variables with arbitrary coefficients

$$q_m = \sum_{i,j=1}^{m} a_{ij}x_ix_j \quad (i \leqq j).$$

Operating with $E_m^{(1)}$, we obtain the polars

$$(4)\qquad \begin{aligned} E_m^{(1)}q_m &= \sum_{i<j=1}^{m} a_{ij}(x_ix_j^p + x_jx_i^p) + 2\sum_{i=1}^{m} a_{ii}x_i^{p+1}, \\ \tfrac{1}{2}E_m^{(1)^2}q_m &= \sum_{i,j=1}^{m} a_{ij}x_i^px_j^p. \end{aligned}$$

These are both formal-modular covariants of q_m under (1). A direct extension shows that if $n \not\equiv 0 \pmod p$, an m-ary form f_n has n covariant polars $E_m^{(r)}f_n$ $(r = 1, 2, \cdots, n)$ of degree 1 in the coefficients. If $p > n$, none of these polar covariants will vanish modulo p. Thus if $n = 3$, $p > 3$, $m = 2$, we have

$$E_2^{(t)} = x_1^{p^t}\frac{\partial}{\partial x_1} + x_2^{p^t}\frac{\partial}{\partial x_2}, \quad f = a_0x_1^3 + 3a_1x_1^2x_2 + \cdots$$

and

$$\begin{aligned} \tfrac{1}{3}E_2^{(t)}f &= a_0x_1^{p^t+2} + 2a_1x_1^{p^t+1}x_2 + a_2x_1^{p^t}x_2^2 + a_1x_1^2x_2^{p^t} \\ &\qquad + 2a_2x_1x_2^{p^t+1} + a_3x_2^{p^t+2}, \\ \tfrac{1}{6}E_2^{(t)^2}f &= a_0x_1^{2p^t+1} + a_1x_1^{2p^t}x_2 + 2a_1x_1^{p^t+1}x_2^{p^t} + 2a_2x_1^{p^t}x_2^{p^t+1} \\ &\qquad + a_2x_1x_2^{2p^t} + a_3x_2^{2p^t+1}, \\ \tfrac{1}{6}E_2^{(t)^3}f &= a_0x_1^{3p^t} + 3a_1x_1^{2p^t}x_2^{p^t} + 3a_2x_1^{p^t}x_2^{2p^t} + a_3x_2^{3p^t}. \end{aligned}$$

The analogy which this polar theory presents when compared with the algebraic polar theory is closer than it might appear to be at first sight. For a little consideration will show that it is immaterial whether we operate directly with

$E_m^{(t)}$, as in the illustrations above, or whether we operate with $E(y) = \left(y\frac{\partial}{\partial x}\right)$ the requisite number of times in succession and then set $y_i = x_i^{p^t}$ $(i = 1, \cdots, m)$. The two results will be identical modulo p. The essential difference between the present theory and the algebraic theory is that an algebraic polar becomes the original polarized form when $(y) = (x)$, whereas in the present theory (y) is expressible in terms of (x) in such a way as to give covariants other than the form itself.

§ 2. *Modular Aronhold Operators.*

Let us suppose that f is a form having $\mu + 1$ coefficients. As previously stated, the coefficients a_i' $(i = 0, \cdots, \mu)$ of the transformed of f under (1) are linear forms with integral coefficients in the variables $a_0, a_1, \cdots$, that is

$$(5)\quad a_i' \equiv \xi_i a_0 + \eta_i a_1 + \cdots + \tau_i a_\mu \pmod{p} \quad (i = 0, 1, \cdots, \mu),$$

where $\xi_i, \eta_i, \cdots$ take all values modulo p which are induced by the linear group (1). Thus (5) form a group induced by (1). Under group (5) the following are cogredient to $a_0, a_1, \cdots, a_\mu$:

$$a_0^{p^t}, a_1^{p^t}, \cdots, a_\mu^{p^t} \quad (t \text{ any positive integer}).$$

Hence (§ 1) the following operator, applied to any concomitant of f, gives a formal concomitant modulo p:

$$\delta_\mu^{(t)} = \left(a^{p^t}\frac{\partial}{\partial a}\right) \equiv a_0^{p^t}\frac{\partial}{\partial a_0} + a_1^{p^t}\frac{\partial}{\partial a_1} + \cdots + a_\mu^{p^t}\frac{\partial}{\partial a_\mu} \pmod{p}.$$

Consider a binary quadratic form

$$f = a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2.$$

Let $p = 3$. The algebraic concomitants are f and its discriminant D, and we have

$$\begin{aligned} \delta_2^{(1)}f &\equiv a_0^3x_1^2 + 2a_1^3x_1x_2 + a_2^3x_2^2 \\ \delta_2^{(1)}D &\equiv a_0^3a_2 + a_1^4 + a_0a_2^3 \end{aligned} \pmod{3}.$$

The latter are formal concomitants, modulo 3, of f. The modular Aronhold operator $\delta_\mu^{(t)}$ applied successively to a concomitant of f

$$\varphi(a_0, a_1, \cdots, a_m, x_1, \cdots)$$

gives a series of formal modular concomitants intermediate, in the sense of Boole, to φ and

$$\varphi_t = \varphi(a_0^{p^t}, a_1^{p^t}, \cdots, a_m^{p^t}, x_1, \cdots). \tag{6}$$

If $b_0, b_1, \cdots, b_\mu$ are the coefficients of a second m-ary quantic g of the same order as f, and φ an invariant function of f and g, then

$$\left(a^{p^t}\frac{\partial}{\partial b}\right), \quad \left(b^{p^t}\frac{\partial}{\partial a}\right),$$

applied to φ, give simultaneous formal modular concomitants.

§ 3. *Modular Transvectants.*

We define the modular transvectant o vo binary forms

$$f(x) = a_0x_1^m + ma_1x_1^{m-1}x_2 + \cdots, \quad \varphi(x) = b_0x_1^n + nb_1x_1^{n-1}x_2 + \cdots$$

$$m, n \not\equiv 0 \pmod p,$$

as follows: Operate upon $f(x)\varphi(y)$ with

$$\Omega = \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}$$

r times, divide by $m\,!\,n\,!/(m-r)\,!\,(n-r)\,!$ and in the result set $y_i = x_i^{p^t}$ $(i = 1, 2)$. The result, which we abbreviate as $(f, \varphi)_{p^t}^r$, is the rth modular transvectant of f and φ. Thus, if $m = n = 2$,

$$(f, \varphi)_p' \equiv (a_0b_1 - a_1b_0)x_1^{p+1} + (a_0b_2 - a_1b_1)x_1x_2^p + (a_1b_1 - a_2b_0)x_1^px_2 + (a_1b_2 - a_2b_1)x_2^{p+1} \pmod p,$$

$$(f, (f, \varphi)_p')_p' \equiv (a_0a_2 - a_1^2)[b_0x_1^{p^2+1} + b_1(x_1^{p^2}x_2 + x_1x_2^{p^2}) + b_2x_2^{p^2+1}]$$

$$\equiv \tfrac{1}{2}D \cdot E_2^{(2)}\varphi(x) \pmod p.$$

I proceed to the problem of finding a canonical formula for $(f, \varphi)_{p^t}^r$, from which several properties can be derived. A well-known form of Gordan's series* gives the expansion of

$$\frac{(m-r)\,!\,(n-r)\,!}{m!\,n!}\,\Omega^r f(x)\varphi(y)$$

as a power series in the argument (xy). In the Aronhold

* Grace and Young, Algebra of Invariants, p. 55.

symbolical notation $(f(x) = a_x^m,\ \varphi(y) = b_y^n)$ this is

$$(ab)^r a_x^{m-r} b_y^{n-r} = \sum_{i=0}^{m-r} \frac{\binom{m-r}{i}\binom{n-r}{i}}{\binom{m+n-2r-i+1}{i}} (f, \varphi)_{y^{n-r}x^{m-r}}^{i+r} (xy)^i.$$

The algebraical transvectant $(f, \varphi)^r$ is obtained from this by the change $y = x$, the right hand side reducing to the first term since $(xx) = 0$. The modular transvectant is obtained from this same expansion by the substitution $y = x^{p^t}$. It does not reduce to a single term on the right but becomes a polynomial in the universal formal modular covariant

$$L_t = (x^{p^t} x) = x_1^{p^t} x_2 - x_1 x_2^{p^t}.$$

THEOREM 2:

$$(f, \varphi)_{p^t}^r \equiv \sum_{i=0}^{m-r} N_i \frac{\binom{m-r}{i}\binom{n-r}{i}}{\binom{m+n-2r-i+1}{i}} E_2^{(t)^{n-i-r}} (f, \varphi)^{i+r} L_t^i,$$

where

$$N_i = (-1)^i (m-i-r)\,!\,/\,[m+n-2(i+r)]\,.\,!$$

In this expansion $E_2^{(t)}$ and L_t are modular and $(f, \varphi)^*$ is algebraic.

We may note that a modular transvectant of a form with itself, of odd index $(f, f)_{p^t}^{2k+1}$, does not vanish identically. But it is reducible in all cases and contains the factor L_t. The transvectants $(f, \varphi)_{p^t}^r$, $(\varphi, f)_{p^t}^r$ are entirely distinct, although they may be called conjugates owing to their symmetrical relationship.

A modular transvectant is a linear combination of modular polars of algebraic transvectants with the universal covariant L_t added to the system. It is known* that L_t is rationally expressible in terms of L_1 and $Q = L_2/L_1$. Moreover I have proved in another paper that Q is a covariant of L_1. Hence one method of procedure in constructing systems of concomitants of a quantic f (modulo p) is to construct the algebraical fundamental system of f and polarize it by the operators $E_2^{(t)}$, $\delta_n^{(t)}$. Then join L_1 to the polar system and form a second system consisting of the simultaneous fundamental system of f and L_1. The forms of the second system which are not

* Dickson, *Transactions*, vol. 12 (1911), p. 75.

found in the polar system are to be added to the polar system. That some forms of the second system will be polars is evident from the fact that, if F is any form whose order is not divisible by p,

$$(L_1, F) = x_1^p \frac{\partial F}{\partial x_1} + x_2^p \frac{\partial F}{\partial x_2} = E_2^{(1)} F \pmod p.$$

§ 4. *Concomitants of a Linear Form.*

Let $f = a_0 x_1 + a_1 x_2$; $p = 3$. The algebraical system of f is f itself. Polarizing this, we have

$$C = E_2^{(1)} f = a_0 x_1^3 + a_1 x_2^3, \quad C' = E_2^{(2)} f = a_0 x_1^9 + a_1 x_2^9,$$
$$D = \delta_1^{(1)} f = a_0^3 x_1 + a_1^3 x_2.$$

The modular system of L_1 is

$$L_1 = x_1^3 x_2 - x_1 x_2^3, \; Q = x_1^6 + x_1^4 x_2^2 + x_1^2 x_2^4 + x_2^6.$$

The simultaneous system of f and L_1 is

$$(L_1, f^r)^r \quad (r = 1, \cdots, 4); \; (Q, f^s)^s \quad (s = 1, \cdots, 6).$$

Of these, some belong to the polar system and some are reducible; as $(Q, f^2)^2 \equiv fC \pmod 3$, etc. But

$$A = (L_1, f^4)^4 \equiv a_0^3 a_1 - a_0 a_1^3,$$
$$B = (Q, f^6)^6 \equiv a_0^6 + a_0^4 a_1^2 + a_0^2 a_1^4 + a_1^6,$$
$$E = (Q, f^3)^3 \equiv a_1(a_0^2 - a_1^2) x_1^3 - a_0^3 x_1^2 x_2 + a_1^3 x_1 x_2^2 + a_0(a_0^2 - a_1^2) x_2^3.$$

The polars

$$E_2^{(1)} D \equiv f^3, \quad E_2^{(1)} E \equiv D L_1, \quad \delta_1^{(1)} A \equiv 0, \quad \delta_1^{(1)} B \equiv A^2 \pmod 3$$

are reducible. The polar C' is also reducible. In fact,

$$C' \equiv CQ - f L_1^2 \pmod 3.$$

The complete set of irreducible concomitants, modulo 3, of the linear form f is

$$A, B, C, D, E, f, L_1, Q.$$

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.

INVARIANTS, SEMINVARIANTS, AND COVARIANTS OF THE TERNARY AND QUATERNARY QUADRATIC FORM MODULO 2.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, September 8, 1914.)

1. A SIMPLE and complete theory of seminvariants of a binary form modulo p was given in the writer's second lecture at the Madison Colloquium.* A fundamental system of covariants of a ternary quadratic form F modulo 2 was obtained in the fourth lecture. In place of the method employed there (pages 77–79) to obtain the leading coefficient of a covariant of F, we shall now present a simpler method which makes it practicable to treat also the corresponding question for quaternary quadratic forms. The new method is, moreover, in closer accord with the underlying principle of those lectures, viz., to place the burden of the determination of the modular invariants upon the separation of the ground forms into classes of forms equivalent under linear transformation. By making the utmost use of this principle, we shall obtain a simpler solution of the problem for the ternary case and then treat the new quaternary case.

Let the coefficients of the quadratic form

$$q_n = \Sigma b_i x_i^2 + \Sigma c_{ij} x_i x_j \quad (i, j = 1, \cdots, n;\ j > i)$$

be undetermined integers taken modulo 2. In a covariant of order ω of q_n, the coefficient of x_n^ω is called the leader and also a seminvariant. It is invariant with respect to the group G generated by the linear transformations on $x_1, \cdots, x_{n-1}$ and those replacing x_n by $x_n + l$, where l is a linear function of $x_1, \cdots, x_{n-1}$, the coefficients in each case being integers taken modulo 2.

2. For $n = 2$, G is composed of the transformations

$$x_1 = x_1', \quad x_2 = x_2' + tx_1'.$$

Taking $t = b_1$ and applying the transformation to

* American Mathematical Society Colloquium Lectures, volume IV, New York, 1914; cited later as Lectures.

$$q_2 = b_1x_1^2 + b_2x_2^2 + c_{12}x_1x_2,$$

we get

$$q_2' = \sigma x_1'^2 + b_2x_2'^2 + c_{12}x_1'x_2',$$

where

$$\sigma = b_1(b_2 + c_{12} + 1).$$

Evidently σ, b_2, and c_{12} are seminvariants. Since therefore they completely characterize the classes of forms q_2 under G, they form a fundamental system of seminvariants of q_2. Since c_{12} and

$$J_2 = (b_1 + c_{12} + 1)(b_2 + c_{12} + 1)$$

remain unaltered when b_1 and b_2 are interchanged, they are invariants and, in fact, form a fundamental system of invariants of q_2.

3. For $n = 3$, we have the seminvariants b_3 and

$$P = (c_{13} + 1)(c_{23} + 1),$$

and the invariants (Lectures, pages 69, 74)

$$A = P(c_{12} + 1), \quad \Delta = c_{12}c_{13}c_{23} + b_1c_{23} + b_2c_{13} + b_3c_{12},$$

$$J_3 = \beta_1\beta_2\beta_3,$$

of which Δ is the discriminant of q_3, and

$$\beta_1 = b_1 + (c_{12} + 1)(c_{13} + 1), \quad \beta_2 = b_2 + (c_{12} + 1)(c_{23} + 1),$$

$$\beta_3 = b_3 + P.$$

THEOREM. *A fundamental system of seminvariants of q_3 is given by b_3, P, A, Δ, J_3.*

It suffices to prove that they completely characterize the classes of forms q_3 under the group G. We have

$$q_3 = q_2 + lx_3 + b_3x_3^2, \quad l \equiv c_{13}x_1 + c_{23}x_2.$$

(I) $b_3 = P = 0$. Then l is not identically zero and can be transformed within G into x_2. In $q_2 + x_2x_3$ we replace x_3 by $x_3 + c_{12}x_1 + b_2x_2$ and obtain $\Delta x_1^2 + x_2x_3$.

(II) $b_3 = 0$, $P = 1$. Then $l \equiv 0$, $q_3 = q_2$. Thus $A = c_{12} + 1$ and $J_3 = J_2$, which completely characterize the classes q_2 (§ 2).

(III) $b_3 = 1$, $P = 1$. In $q_3 = q_2 + x_3^2$ we replace x_3 by $x_3 + b_1x_1 + b_2x_2$ and obtain $c_{12}x_1x_2 + x_3^2$, where $c_{12} = A + 1$.

(IV) $b_3 = 1$, $P = 0$. As in (I), we may set $l = x_2$. Re-

placing x_3 by $x_3 + b_1x_1$, we may set also $b_1 = 0$ in q_3. Then $\Delta = c_{12}$, $J_3 = (c_{12} + 1)b_2$. Thus, if $\Delta = 0$, $q_2 = J_3x_2^2$. But, if $\Delta = 1$, we replace x_1 by $x_1 + b_2x_2$ and get $q_2 = x_1x_2$. In either case, the coefficients of the final q_3 are determined by the seminvariants.

COROLLARY. *There are exactly eleven linearly independent seminvariants of q_3; they may be taken to be* 1, A, Δ, J_3, AJ_3, b_3, b_3A, $b_3\Delta$, b_3J_3, P, b_3P.

In fact, the number of classes in the four cases was 2, 4, 2, 3, respectively. Hence (Lectures, page 13) there are exactly 11 linearly independent seminvariants. The 11 functions in the corollary can be proved to be linearly independent either in the usual direct manner or more simply by noting that any polynomial in b_3, P, A, Δ, J_3 can be reduced modulo 2 to a linear function of the 11 by means of the relations

$$\Delta A = \Delta J_3 = b_3AJ_3 = 0, \quad PA = P,$$

$$P\Delta = b_3(A + P), \quad PJ_3 = (b_3 + 1)J_3.$$

4. For $n = 4$, the discriminant of q_n is the Pfaffian

$$[1234] = c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23}.$$

Another evident invariant of q_4 is $A_4 = Au$, where

$$u = (c_{14} + 1)(c_{24} + 1)(c_{34} + 1).$$

We shall employ the abbreviations

$$B_1 = b_1 + (c_{12} + 1)(c_{13} + 1)(c_{14} + 1),$$

$$B_2 = b_2 + (c_{12} + 1)(c_{23} + 1)(c_{24} + 1),$$

$$B_3 = b_3 + (c_{13} + 1)(c_{23} + 1)(c_{34} + 1),$$

$$B_4 = b_4 + u.$$

Then* q_4 has the further invariants

$$J_4 = B_1B_2B_3B_4 + b_1b_2b_3b_4[1234],$$

$$\begin{aligned} K = {} & \Sigma b_1b_2c_{34} + \Sigma b_1(c_{23}c_{24} + c_{23}c_{34} + c_{24}c_{34} + c_{23} + c_{24} + c_{34}) \\ & + [1234]\Sigma b_1 + \Sigma c_{12} + \Sigma c_{12}c_{13} + \Sigma c_{12}c_{13}c_{14} \\ & + c_{12}c_{13}c_{24}c_{34} + c_{12}c_{14}c_{23}c_{34} + c_{13}c_{14}c_{23}c_{24} + \Sigma c_{12}c_{13}c_{24}, \end{aligned}$$

* *Proc. Lond. Math. Soc.*, ser. 2, vol. 5 (1907), p. 308.

the subscripts in the final sum being

12 13 24, 12 13 34, 12 14 23, 12 14 34,
13 14 23, 13 14 24, 12 24 34, 12 23 34,
13 23 24, 13 24 34, 14 23 24, 14 23 34.

We have the seminvariants b_4 and u since

$$q_4 = q_3 + \lambda x_4 + b_4 x_4^2, \ \lambda = \sum_{i=1}^{3} c_{i4} x_i.$$

THEOREM. *A fundamental system of seminvariants of q_4 is given by b_4, u, [1234], A_4, J_4, K.*

We prove that they characterize the classes q_4 under G.

(I) $b_4 = u = 0$. Then λ is not identically zero and can be transformed within G into x_3. In $q_3 + x_3x_4$ we replace x_4 by $x_4 + c_{13}x_1 + c_{23}x_2 + b_3x_3$ and get $x_3x_4 + q_2$. Then $c_{12} = [1234]$ and $J_2 = K$ characterize the resulting classes (§ 2).

(II) $b_4 = 0$, $u = 1$. Then $\lambda \equiv 0$, $q_4 = q_3$, and

$$A_4 = A, \quad J_4 = J_3, \quad K = \Delta + A + 1$$

form a fundamental system of invariants of q_3 (§ 3).

(III) $b_4 = u = 1$. If $A_4 = 1$, $q_4 = \Sigma b_i x_i^2 + x_4^2$. Replacing x_4 by $x_4 + \Sigma b_i x_i$, we get x_4^2. If $A_4 = 0$, we may take $c_{12} = 1$, replace x_1 by $x_1 + c_{23}x_3$, x_2 by $x_2 + c_{13}x_3$ and have $c_{13} = c_{23} = 0$. Replacing x_4 as before, we get $x_1x_2 + x_4^2$.

(IV) $b_4 = 1, u = 0$. We may set $\lambda = x_3$ as in (I). Replacing x_4 by $x_4 + c_{13}x_1 + c_{23}x_2$, we have $c_{13} = c_{23} = 0$. Then $[1234] = c_{12}$.

First, let $c_{12} = 0$. Then $K = (b_1 + 1)(b_2 + 1)$, $J_4 = b_3K$. If $K = 1$, then $b_1 = b_2 = 0, b_3 = J_4$, and q_4 is fixed. If $K = 0$, we may set $b_1 = 1$, replace x_1 by $x_1 + b_2x_2 + b_3x_3$ and get $b_1 = 1$, $b_2 = b_3 = 0$.

Second, let $c_{12} = 1$. Then $J_4 = 0, K = b_1b_2 + b_3$. Replace x_1 by $x_1 + b_2x_3$, x_2 by $x_2 + b_1x_3$, x_4 by $x_4 + b_1x_1 + b_2x_2$. We get

$$x_1x_2 + Kx_3^2 + x_3x_4 + x_4^2.$$

COROLLARY. *There are exactly sixteen linearly independent seminvariants of q_4; they may be taken to be the invariants*

$$1, \ k = [1234], \ A_4, \ J_4, \ K, \ A_4J_4, \ kK,$$

the products of b_4 by the preceding other than A_4J_4, and

$$u,\ b_4u,\ (b_4+1)u\Delta.$$

For, the number of classes in the four cases was 4, 5, 2, 5, respectively. Any polynomial in the six seminvariants given in the theorem can be reduced to a linear function of the sixteen in the corollary by use of*

$$kA_4 = kJ_4 = A_4K = 0,\ \ J_4(K+A_4+1) = 0,\ \ b_4A_4J_4 = 0,$$

$$uA_4 = A_4,\ \ uk = 0,\ \ uJ_4 = (b_4+1)J_4,\ \ uK = (b_4+1)(u+A_4+u\Delta).$$

5. Consider a covariant of odd order ω of q_4

$$C = Sx_4^n + S_1x_4^{n-1}x_1 + \cdots.$$

Now $x_4 = x_4' + x_1'$ replaces q_4 by q_4' in which

$$(1)\quad c_{12}' = c_{12} + c_{24},\ \ c_{13}' = c_{13} + c_{34},\ \ b_1' = b_1 + b_4 + c_{14}.$$

If the latter replaces S_1 by S_1', we have $S_1' = S_1 + S$. Hence S has no term with the factor $c_{12}c_{13}b_1$. Of the functions in the corollary, only J_4, A_4J_4, and b_4J_4 contain $c_{12}c_{13}b_1$, and its coefficients in them are linearly independent. Hence

$$S = I + b_4I_1 + cu + db_4u + e(b_4+1)u\Delta,$$

where I and I_1 are linear combinations of 1, k, A_4, K, kK.

From geometrical considerations (Lectures, page 72),

$$L = (k+1)\{(B_1+1)x_1 + \cdots + (B_4+1)x_4\}$$

is a covariant. The coefficients of x_4^n in iL^n $(i = 1, A_4, K)$ are

$$(k+1)(b_4+1)+u,\ A_4b_4,\ (b_4+1)(Kk+K+A_4+u+u\Delta).$$

After subtracting multiples of iL^n from C we may therefore assume that $c = e = 0$ and that I_1 is free of A_4.

First, let $\omega = 1$. Thus S_1 is derived from S by permuting the subscripts 1 and 4. Then $S_1' = S_1 + S$ gives

$$I = c_{14}I_1 + d(c_{14}+1)\{b_1(c_{24}\alpha + c_{12}c_{34} + c_{34}) + b_4(c_{12}\alpha + c_{13}c_{24} + c_{13})\},$$

where $\alpha = c_{13} + c_{34} + 1$. Let Σ denote the sum of the second member and the function obtained from it by interchanging the subscripts 1 and 2. Thus $\Sigma = 0$. Taking $c_{24} = c_{14}$, we

* The first four from the table of the paper last cited, p. 311.

see that $d = 0$. Thus $(c_{14} + c_{24})I_1 = 0$. Apply $x_2 = x_2' + x_1'$, whence

$$c_{13}' = c_{13} + c_{23}, \quad c_{14}' = c_{14} + c_{24}, \quad b_1' = b_1 + b_2 + c_{12}.$$

Then $c_{14}I_1 = 0$. Hence every $c_{ij}I_1 = 0$, $I_1 = lA_4$. But I_1 is free of A_4. Hence $I_1 = 0$, $I = 0$, $S = 0$.

THEOREM. *Every linear covariant of q_4 is a linear function of L, A_4L, KL.*

Next, let $\omega > 1$. After subtracting from C a constant multiple of $q_4L^{\omega-2}$, whose leader is b_4u, we have $d = 0$ in S. Express S_1 as a polynomial in c_{12}, c_{13}, b_1, and call p the coefficient of their product. The coefficient of $c_{12}c_{13}$ in $S_1' - S_1 = S$, found from (1), is $p(b_4 + c_{14})$, and hence vanishes if $b_4 = c_{14}$; while S itself vanishes if also $c_{24} = c_{34} = 0$. Applying these two conditions to $S = I + b_4I_1$, we find that

$$S = (b_4 + 1)k(n + mK), \qquad n, m \text{ constants.}$$

Several tests failed to exclude this leader. Whether or not there are covariants with such a leader S is not discussed here.

In this connection, note the covariant

$$\Sigma c_{ij}(x_ix_j^{x'} + x_i^{x'}x_j) \quad (i, j = 1, \cdots, 4;\ i < j),$$

obtained by replacing the variables in the polar of (x) with respect to q_4 by $x_k^{x'}$ $(k = 1, \cdots, 4)$.

6. By means of the corollary in § 4, and transformation (1), we readily obtain the

THEOREM. *Every quadratic covariant of q_4 is a linear function of I^2, KI^2, Iq_4, where I is an invariant.*

UNIVERSITY OF CHICAGO,
June, 1914.

THE CONVERSE OF THE HEINE-BOREL THEOREM IN A RIESZ DOMAIN.

BY DR. E. W. CHITTENDEN.

(Read before the American Mathematical Society, April 11, 1914.)

IN various generalized forms of the Heine-Borel theorem*

* Cf. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, vol. 22 (1906), p. 26; and T. H. Hildebrandt, "A contribution to the foundations of Fréchet's calcul fonctionnel," *Amer. Jour. of Mathematics*, vol. 34 (1912), p. 282.

for a class $\mathfrak{Q}$ of elements q, the condition that $\mathfrak{Q}$ be extremal* appears as a necessary condition. It is the purpose of the present paper to show that this condition is necessary in any domain for which a Fréchet limit is defined; and furthermore, that if the property extremal be replaced by the property self-compact (a class is self-compact if its every infinite subclass has a limiting element in the class), more general domains may be considered,—notably; those of F. Riesz† for which limiting element is the fundamental relation, and those of T. H. Hildebrandt‡ for which a limit of a sequence need not be unique.§

1. *Self-Compact Classes in a Riesz Domain.*

A Riesz domain is formed by a class $\mathfrak{P}$ which admits a relation "limiting element" between single elements and subclasses $\mathfrak{T}$ of $\mathfrak{P}$ with the following properties:

(1) If $\mathfrak{T}$ has a limiting element in $\mathfrak{P}$, $\mathfrak{T}$ contains at least two elements.

(2) If p is a limiting element of $\mathfrak{T}$, p is a limiting element of any class containing $\mathfrak{T}$.

(3) If p is a limiting element of $\mathfrak{T}$, and $\mathfrak{T}$ is divided into classes $\mathfrak{T}_1$, $\mathfrak{T}_2$, then p is a limiting element of $\mathfrak{T}_1$ or $\mathfrak{T}_2$.‖

In a Riesz domain an element p is interior $(\mathfrak{R})$¶ to $\mathfrak{S}$ in case $\mathfrak{S}$ contains p and a single element (distinct from p) of every class $\mathfrak{T}$ of $\mathfrak{R}$ which has p for a limiting element.

A family $[\mathfrak{S}]$ of classes $\mathfrak{S}$ is an extended enclosure of a class $\mathfrak{Q}$ if for every element q of $\mathfrak{Q}$ there exists an $\mathfrak{S}_q$ of $\mathfrak{S}$ such that q is interior $(\mathfrak{Q})$ to $\mathfrak{S}$.

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* Cf. Fréchet, loc. cit., p. 7.

† "Stetigkeitsbegriff und abstrakte Mengenlehre," Atti del IV Congresso Internazionale dei Matematiche, Roma, 1908, vol. 2, pp. 18–24.

‡ Loc. cit., p. 241.

§ It is desired to call attention to the fact that it is self-compactness rather than extremality which is essential in many theorems of the calcul fonctionnel of Fréchet, and that by the use of this concept most of the theorems of Fréchet which involve the hypothesis of extremality may be extended to domains for which the limit relation is not subject to the uniqueness condition. This has been done to a large extent for classes which admit a development Δ in a paper presented to the American Mathematical Society in 1913 by Professor Pitcher and Dr. Chittenden, which is not yet published.

‖ The three properties just given are equivalent to the ones stated by F. Riesz in his Rome paper already cited. That $\mathfrak{T}$ contains an infinity of elements if it has a limiting element is a consequence of (1) and (3). This is established by successive removals of a single element from $\mathfrak{T}$. There always must remain two at least.

¶ Read: interior to $\mathfrak{S}$ relative to $\mathfrak{R}$. This relativity feature was introduced by T. H. Hildebrandt, loc. cit., p. 268 (10).

A class $\mathfrak{Q}$ has the Heine-Borel property if every family $[\mathfrak{S}]$ which is an extended enclosure of $\mathfrak{Q}$ contains a *finite* subfamily with the same property.

THEOREM I. *If a class $\mathfrak{Q}$ in a Riesz domain has the Heine-Borel property, $\mathfrak{Q}$ is self-compact.*

We have to show that every infinite subclass of $\mathfrak{Q}$ has a limiting element in $\mathfrak{Q}$. Suppose there exists an infinite subclass $\mathfrak{R}$ of $\mathfrak{Q}$ with no limiting element in $\mathfrak{Q}$. From condition (2) it follows that $\mathfrak{R}$ contains a sequence $\{q_n\}$ of distinct elements with no limiting element in $\mathfrak{Q}$. We define $[\mathfrak{S}_n]$, an extended enclosure of $\mathfrak{Q}$, as follows: $\mathfrak{S}_n$ contains all elements of $\mathfrak{Q}$ except the elements $q_{n+1}, q_{n+2}, \cdots$. Every element of $\mathfrak{Q}$ is contained in some $\mathfrak{S}_n$. Let q be any element of $\mathfrak{Q}$ and $\mathfrak{T}$ any subclass of $\mathfrak{Q}$ which has q for a limiting element. Let $\mathfrak{T}_1$ denote the elements of $\mathfrak{T}$ which are in the sequence $\{q_n\}$, $\mathfrak{T}_2$ the remaining elements of $\mathfrak{T}$. By (2) q is not a limiting element of $\mathfrak{T}_1$, and by (3) therefore, q is a limiting element of $\mathfrak{T}_2$. But $\mathfrak{T}_2$ is contained in every $\mathfrak{S}_n$. Hence $[\mathfrak{S}_n]$ is an extended enclosure of $\mathfrak{Q}$, such that no finite subfamily of $[\mathfrak{S}_n]$ contains all the elements of $\mathfrak{Q}$. This is contrary to hypothesis, and the proposition is proved.*

2. *Limiting Element Defined in Terms of Limit. Systems* $(\mathfrak{P}; L)$.

Following T. H. Hildebrandt,† we denote by L (limit) any relation between a sequence of elements and a single element. If a relation L holds between a sequence $\{p_n\}$ and an element p, we say p is a limit of $\{p_n\}$ and write

$$p = L_n p_n.$$

A class $\mathfrak{P}$ and a relation L defined for that class form a system $(\mathfrak{P}; L)$.

The following properties are fundamental in the theory of the relation L:

(1) If a sequence has a limit, the limit is unique.

(2) If a sequence has a limit, its every subsequence, taken in the same order, has the same limit.

(3) The identical sequence $\{p_n\}$, $p_n = p(n)$, has p for a limit. A relation L having a property i ($i = 1, 2, 3, 12, 13, 23, 123$) will be denoted by L^i. An L^{123} is a Fréchet limit.‡

* If $\mathfrak{Q}$ is finite it is compact in a vacuous sense. Every finite class has the Heine-Borel property.

† Loc. cit., p. 241.

‡ Loc. cit., pp. 6–7.

In a system $(\mathfrak{P}; L)$, p is a limiting element of class $\mathfrak{T}$ if $\mathfrak{T}$ contains a sequence $\{p_n\}$ of distinct elements having p for a limit.

The following theorem follows readily from this definition of limiting element.

THEOREM II. *In a system* $(\mathfrak{P}; L^2)$, $\mathfrak{P}$ *is a Riesz domain.*

As an immediate consequence of theorems I and II we have:

THEOREM III. *In a system* $(\mathfrak{P}; L^2)$, *every class* $\mathfrak{Q}$ *which has the Heine-Borel property is self-compact.*

We may extend the result of Theorem III with the aid of the following lemma.

LEMMA. *In a system* $(\mathfrak{P}; L^{12})$, *every self-compact class* $\mathfrak{Q}$ *is extremal.*

$\mathfrak{Q}$ is compact by hypothesis. We have to show that $\mathfrak{Q}$ is closed. That is, if p is a limiting element of $\mathfrak{T}$, a subclass of $\mathfrak{Q}$, then p belongs to $\mathfrak{Q}$. If p is a limiting element of $\mathfrak{T}$ then a sequence $\{q_n\}$ of distinct elements of $\mathfrak{T}$ exists which has p for a limit. As an infinite class $\{q_n\}$ has a limiting element q in $\mathfrak{Q}$ because $\mathfrak{Q}$ is self-compact. $\{q_n\}$ therefore contains a subsequence $\{q_{n_i}\}$ of distinct elements with limit q. $\{q_{n_i}\}$ contains a subsequence $\{q_{n_{k_m}}\}$ whose order is the same as that of both $\{q_n\}$ and $\{q_{n_i}\}$. Therefore by L^2, $q = L_m q_{n_{k_m}}$ and $p = L_m q_{n_{k_m}}$. By L^1, q and p are the same and therefore p belongs to $\mathfrak{Q}$; which was to be shown. We have from Theorem III and the lemma

THEOREM IV. *In a system* $(\mathfrak{P}; L^{12})$, *every class* $\mathfrak{Q}$ *which has the Heine-Borel property is extremal.*

It will be noticed that the character of the result in Theorem IV is determined by the definition of interior given in §1 for Riesz domains. We will show that in a system $(\mathfrak{P}; L^2)$ this definition is (except for relativity) equivalent to the definition used (implicitly) by Fréchet in the theorem cited at the beginning of this paper. It will follow that Theorem IV includes as a special case the result of Fréchet.

The definition of interior used by Fréchet (except for relativity) is equivalent to the following: q is interior $(\mathfrak{R})$ to $\mathfrak{S}$ if $\mathfrak{S}$ contains q and every sequence $\{r_n\}$ of distinct elements of $\mathfrak{R}$ which has q for a limit is ultimately* contained in $\mathfrak{S}$.

* In the sense: there exists n_0 such that $n \geqq n_0$ implies r_n is contained in S.

If now p is Fréchet interior to $\mathfrak{S}$ and is a limiting element of $\mathfrak{T}$ a subclass of $\mathfrak{R}$, then, by definition of Fréchet interior, limiting element, and I^2, $\mathfrak{S}$ contains an infinity of elements of $\mathfrak{T}$. Therefore p is interior to $\mathfrak{S}$ in the sense of § 1. Furthermore if p is interior ($\mathfrak{R}$) to $\mathfrak{S}$ in the sense of § 1, then $\mathfrak{S}$ contains an element q (distinct from $\mathfrak{P}$) of every subclass $\mathfrak{T}$ of $\mathfrak{R}$ for which p is a limiting element. Then if $p = L_n r_n$ (distinct) p is a limiting element of the class $[r_n]$. Hence $\mathfrak{S}$ contains r_{n_1} distinct from p. Since p is a limiting element of the class obtained from $[r_n]$ by removing r_{n_1} (I^2) it is evident that at most a finite number of elements of $[r_n]$ are not in $\mathfrak{S}$. Therefore $[r_n]$ is ultimately contained in $\mathfrak{S}$.

T. H. Hildebrandt* has given a definition of interior ($\mathfrak{R}$) which becomes equivalent to the Fréchet interior ($\mathfrak{R}$) for systems ($\mathfrak{P}$; I^{123}). This definition omits the condition that the sequence $\{r_n\}$ consist of distinct elements. If then $p = L_n r_n$ and r_{n_0} is repeated infinitely often, in a system ($\mathfrak{P}$; I^{123}), $r_{n_0} = p$. That r_{n_0} is contained in any class $\mathfrak{S}$ to which p is Fréchet interior ($\mathfrak{R}$) is evident. A restatement of Theorem IV for systems ($\mathfrak{P}$; I^{123}) gives us a generalization of a theorem of Hildebrandt.†

URBANA, ILL.,
October 28, 1914.

COMPLETE EXISTENTIAL THEORY OF SHEFFER'S POSTULATES FOR BOOLEAN ALGEBRAS.

BY PROFESSOR L. L. DINES.

(Read before the American Mathematical Society, December 30, 1913.)

IN a recent number of the *Transactions* Sheffer‡ presented an elegant and concise set of five postulates for Boolean algebras, and proved them mutually consistent and independent. Professor E. H. Moore§ has suggested a further interesting problem in connection with such sets of postulates, namely the determination of all general implicational relations

* Loc. cit., p. 268 (10).

† Loc. cit., p. 282 (2).

‡ H. M. Sheffer, "A set of five postulates for Boolean algebras with application to logical constants," *Transactions*, vol. 14 (1913), pp. 481–488.

§ E. H. Moore, "Introduction to a form of general analysis," New Haven Mathematical Colloquium, Yale University Press, page 82.

which exist between properties defined either by the postulates themselves or by the negatives of the postulates. A set of postulates are then said to be *completely independent*, if and only if no such implicational relations exist. For example, though Sheffer's postulates are independent in the ordinary sense that no four of them imply a fifth, they are not completely independent, for it can be shown that the negative of the first postulate implies the third, fourth, and fifth.

The postulates in question define properties of a system consisting of a class $\mathfrak{K}$ of undefined elements and an undefined binary rule of combination $|$ between elements of $\mathfrak{K}$. Any system $\Sigma(\mathfrak{K}, |)$ of the prescribed type has, with respect to the five properties defined by the postulates, one of the $2^5 = 32$ characters:

$$(+++++),\ (-++++),\ \cdots,\ (----+),\ (-----); \tag{1}$$

the ith sign of the character being $+$ or $-$ according as Σ has or has not the ith property. The body of $2^5 = 32$ propositions stating for the various characters (1) that there does or does not exist a system having the character in question, constitutes what Professor Moore has called "the complete existential theory" of the five postulates.

§ 1. *Sheffer's Postulates Concerning a System* $\Sigma(\mathfrak{K}, |)$.

The five postulates of Sheffer are:

1. There are at least two elements in $\mathfrak{K}$.

2. Whenever a and b are elements of $\mathfrak{K}$, $a \mid b$ is an element of $\mathfrak{K}$.

Definition. $a' = a \mid a$.

3. Whenever a and the indicated combinations of a are elements of $\mathfrak{K}$,

$$(a')' = a.$$

4. Whenever a, b, and the indicated combinations of a and b are elements of $\mathfrak{K}$,

$$a \mid (b \mid b') = a'.$$

5. Whenever a, b, c, and the indicated combinations of a, b, and c are elements of $\mathfrak{K}$,

$$[a \mid (b \mid c)]' = (b' \mid a) \mid (c' \mid a).$$

In what follows, the fact that a system Σ has the property defined by the ith postulate will be indicated* by placing i as a superscript to Σ. The fact that Σ does not have the property defined by the ith postulate will be denoted by placing $-i$ as a superscript to Σ. For example we may express the facts that Σ satisfies postulates 2 and 3, but does not satisfy postulate 4 by Σ^{23-4}.

§ 2. *Complete Existential Theory.*

THEOREM. *For the five postulates 1–5 concerning systems* $\Sigma(\mathfrak{K}, |)$, *the complete existential theory consists of 14 propositions of non-existence, and 18 propositions of existence. In particular the four postulates 2–5 are completely independent. The non-existences are expressed by the proposition* †

$$\Sigma^{-1} \supset \Sigma^{345}. \tag{2}$$

To prove proposition (2) we need only note that the hypothesis necessitates either $\mathfrak{K}^{\text{null}}$ (that is that $\mathfrak{K}$ have no elements) or $\mathfrak{K}^{\text{singular}}$ (that is that $\mathfrak{K}$ have only one element). In the former case Σ satisfies postulates 3, 4, 5 vacuously. In the latter case Σ satisfies postulates 3, 4, 5 either vacuously or evidently, according as Σ does not or does satisfy postulate 2.

Proposition (2) renders impossible the existence of systems Σ with the following 14 characters:

$$(-+-++),\ (-++-+),\ (-++++-),\ (-++--),$$
$$(-+-+-),\ (-+--+),\ (-+---),$$
$$(---++),\ (--+-+),\ (--++-),\ (--+--),$$
$$(---+-),\ (----+),\ (-----).$$

We next give examples of systems Σ having each of the other 18 characters as follows: two examples for $\mathfrak{K}^{\text{singular}}$, ten examples for $\mathfrak{K}^{\text{dual}}$, and six examples for $\mathfrak{K}^{\text{triple}}$. In each case the class $\mathfrak{K}$ has the smallest number of elements possible.

* The scheme of notation here described is used by Professor Moore in his General Analysis.

† The symbol $\supset$ used in the statement of this proposition is the symbol of logical implication used by Professor Moore and others. In words, proposition (2) may be stated: "If a system Σ does not satisfy postulate 1, then Σ does satisfy postulates 3, 4, and 5."

Examples for $\mathfrak{K}$ singular.

A class $\mathfrak{K}$ with single element m furnishes systems Σ with the following two characters:

$(-++++)$ when $m / m = m$,

$(--+++)$ when $m / m \neq m$.

In the examples for $\mathfrak{K}$ dual, and $\mathfrak{K}$ triple, the operation $|$ will be defined by means of tables. For instance if $\mathfrak{K}$ has two elements m and n; and if $m \mid m = n$, $m \mid n = m$, $n \mid m = n$, and $n \mid n$ is not an element of $\mathfrak{K}$, this will be expressed by the table

$\mid$	m	n
m	n	m
n	n	—

Examples for $\mathfrak{K}$ dual.

$(+++++)$

$\mid$	m	n
m	n	m
n	m	m

$(+--++)$

$\mid$	m	n
m	n	—
n	—	n

$(+-+++)$

$\mid$	m	n
m	m	—
n	—	n

$(+-+-+)$

$\mid$	m	n
m	m	n
n	—	n

$(++-++)$

$\mid$	m	n
m	m	m
n	m	m

$(++-+-)$

$\mid$	m	n
m	n	m
n	m	n

$(+++-+)$

$\mid$	m	n
m	m	n
n	n	n

$(+++--)$

$\mid$	m	n
m	m	n
n	m	n

(++++−)

	m	n
m	m	m
n	n	n

(+−−−+)

	m	n
m	n	m
n	—	n

That there are no examples of other characters to be obtained for $\mathfrak{K}$ dual is shown by the two propositions*

(3) $$\mathfrak{K}^{\text{dual}} \cdot \Sigma^{-3} : \supset : \Sigma^{5}$$

(4) $$\mathfrak{K}^{\text{dual}} \cdot \Sigma^{-2} : \supset : \Sigma^{5},$$

both of which follow without difficulty from the definitions involved.

Examples for $\mathfrak{K}$ triple.

(+−++−)

	l	m	n
l	l	m	n
m	n	n	l
n	m	—	—

(++−−−)

	l	m	n
l	l	m	n
m	n	l	n
n	m	n	m

(+−+−−)

	l	m	n
l	l	m	n
m	l	m	n
n	l	—	n

(+−−+−)

	l	m	n
l	l	m	—
m	l	l	l
n	m	l	m

(++−+−)

	l	m	n
l	l	m	n
m	l	l	l
n	m	l	m

(+−−−−)

	l	m	n
l	l	m	—
m	n	l	m
n	—	—	m

That postulates 2–5 are *completely independent* follows from the fact that systems have been exhibited having the $2^4 = 16$ characters $(+\pm\pm\pm\pm)$.

* Proposition (3): If the class $\mathfrak{K}$ has exactly two elements and the system Σ does not satisfy postulate 3, then Σ does satisfy postulate 5. The interpretation of (4) is similar.

§ 3. *Concerning a Substitute for Postulate 1.*

The fact that postulate 1 prevents the set of five postulates from being completely independent suggests the desirability of replacing it by another postulate such that the resulting set shall be completely independent. Evidently the new postulate must assume as a minimum number of elements, a greater number than *two* in order that proposition (2) shall not hold.

If postulate 1 be replaced by the statement that $\mathfrak{R}$ shall have at least *three* distinct elements, the difficulty is not overcome, for there still exist no systems having the characters

$$(-+-+-),\quad (-+---),\quad (--++-),\quad (--+--),$$
$$(---+-),\quad (-----),$$

as is shown by propositions (3) and (4).

It seems rather evident that a completely independent set of postulates could be obtained by postulating as the minimum number of elements of $\mathfrak{R}$ a sufficiently great number. How great this minimum must be has not however been determined by the author.

UNIVERSITY OF SASKATCHEWAN.

ON THE CHARACTERISTICS OF THE PRINCIPAL MANUALS OF ELEMENTARY GEOMETRY PUBLISHED IN ITALY IN THE COURSE OF THE LAST FIFTY YEARS.*

BY PROFESSOR MARIO VECCHI.

IN the first half of the nineteenth century, the Elements of Geometry by A. M. Legendre dominated instruction in that subject in Italy. But about 1860, need for a better exposition was felt, and Italian mathematicians, on the initiation of Luigi Cremona, began to turn to the limpid clearness of Euclid. In the year 1867, the Italian government ordered that geometry should be taught on euclidean lines in the classical gymnasia, and in the same year the well-known

* Originally published as Supplement No. I of A. R. Kulischer's translation into Russian of J. W. A. Young's Teaching of Mathematics. The English translation here published has been prepared by A. R. Kulischer.

mathematicians, Enrico Betti and Francesco Brioschi, published an edition of the first books of Euclid for school use,* based in the main on the edition of Viviani. But though this antique edifice is beyond reproach in the harmony of its distinct parts and in the critical and creative power displayed therein, in form and style Euclid's Elements are not altogether in conformity with the tastes of our more modern times. Accordingly, the Minister of Education permitted schools to substitute for Euclid's Elements other works preserving the general euclidean scheme and the rigor of the ancient methods. In consequence of this permission, textbooks appeared, retaining the essential features of Euclid's work, and at the same time enriching the contents with other material more suitable to our epoch. We mention the work of Professors Sannice and D'Ovidio,† wherein for the first time the intuitional concepts, used tacitly by Euclid, are given precise formulation, making the *postulate of motion* the corner stone.

The work by Faifofer‡ is distinguished not only by exceptional clearness of exposition but also by considerable value from the critical point of view. Here, for the first time in elementary works, the theory of equivalence is expressed in the purely geometric form, thus meeting the need pointed out by Duhamel. The work uses considerably fewer postulates than are used by earlier writers, treats proportion by the aid of new, clear, and very simple considerations, and for the first time, indicates clearly what results are independent of the parallel postulate.

A third work, which like the two already named makes systematic use of the postulate of motion, is remarkable for the novelty of its views. I refer to the Elements of Geometry by K. de Paolis.§

Adhering, with Hoüel, to the views of Helmholtz, this author, like Faifofer, postulates the motion of rigid figures, but in a form logically more satisfying, and defines figures as equal when they can be brought to coincidence by motion. De Paolis, exhibiting in this work his deep scientific knowledge

* Gli elementi d'Euclide, per cura di E. Betti e F. Brioschi, Florence, 1867.

† Sannice e D'Ovidio, Elementi di Geometria, Naples, 1869.

‡ A. Faifofer, Elementi di Geometria, Venice, 1880. French translation by Fr. Talanti, Paris, 1903.

§ K. de Paolis, Elementi di Geometria, Turin, 1884.

and keen critical sense, became the standard bearer of that considerable group of authors who advocated the fusion of plane and solid geometry. He had shown by an actual example that by rejecting the ancient subdivision of geometry useful simplifications can be made in both parts. The plan of fusion was for a long time looked at askance, but in 1900, official permission was given to use it in the schools after the matter of the first three books of Euclid had been studied. Consequently, a number of textbooks have appeared, deviating from the euclidean system, and embodying the results of modern criticism of the principles of geometry, thus bringing to this subject in the schools a greater measure of precision and rigor.

The tide reached its height with the appearance of the Elements of Geometry by Veronese.* This well-known scholar, basing his work on a profound analysis of principles, had already set forth the leading thoughts of his geometric system in his work on hyperspace.†

In the above named Elements he gives (provisionally in the form of a school text-book) a rigorous exposition of the logical foundations of geometry. After having excluded the idea of motion, so characteristic of the text-books named above, he enumerates explicitly all the postulates on which his edifice is based, appealing unhesitatingly to intuition in the interests of freer thinking. He starts from a single fundamental notion, that of a point, and from this the other geometric forms are developed without further need of existence postulates. The most characteristic feature of this work is its original theory of equality. As fundamental concept, that of congruence of segments is here taken. The author practices fusion to a certain extent, and treats simultaneously for the straight line, the plane, and space certain special questions, such as the theory of equality, of similitude, and mensuration. In a short period Veronese published a series of editions of his text book, and crowned the whole with a set of text books for the various types of secondary schools, conforming to the curricula of 1900. In each of these works, his fine critical feeling and his great learning are clearly exhibited. To the courses of Veronese the Elements of

* G. Veronese, Elementi di Geometria, Padua, 1897.

† G. Veronese, Fondamenti di Geometria a piu Dimensioni, Padua. German translation by A. Schepp. Leipzig, 1894.

Geometry by Ingrami* must be adjoined,w hich independently of the former made analogous contributions to elementary geometry.

In 1900 there was published by F. Enriques† a collection of valuable papers by numerous collaborators on various questions of elementary geometry, dealing in part with the subject matter and in part with problems of teaching. In 1903 F. Enriques and Ugo Amaldi published a work on elementary geometry, in which are combined with admirable harmony, rigorous and scientific exposition of the principles and compliance with pedagogic requirements. The authors, conforming to those previous writers who exclude the principle of motion, adopt the standpoint of Hilbert whose postulates, in their essential points, find a place here. The novelty and rigor of the exposition give value also to the theory of equivalence, which is presented in an unusually complete form. The authors base the correspondence of polygons and prisms on decomposition into equal parts (Duhamel), and for such forms for which this criterion is insufficient (as proved by Rethy, Dehn, and recently Kagan) the concept of "equality of extensions" is introduced, permitting a logically rigorous treatment. In the chapter on proportion, the authors remain on the ground of euclidean definitions, and succeed in reducing the abstract side of the question to a minimum, and in placing the concrete geometric applications naturally in the foreground. From the novel details of the book, there may be selected for mention an exceedingly elegant treatment of the equality of trihedral and polyhedral angles without the use of the parallel postulate. Such are some characteristic features of their excellent work which combines classic clearness of form and perfect limpidity of expression.

We must mention finally the work of De Franchis,‡ written in the manner of Veronese, and making the interesting experiment of introducing a rigorous treatment of the concept of motion into a school text book.

ROME, ITALY.

* G. Ingrami, Elementi di Geometria, Bologna, 1904.

† F. Enriques, Questioni riguardanti la Geometria elementare, Bologna, 1900. German translation: Die Fragen der Elementargeometrie, Leipzig, 1907.

‡ Naples, 1909.

MATHEMATICAL METHODS IN PHYSICS.

Sur quelques Progrès récents de la Physique mathématique. Par VITO VOLTERRA, Clark University Lectures of 1909, published by Clark University, 1912. 82 pp.

Drei Vorlesungen über neuere Fortschritte der mathematischen Physik. Von VITO VOLTERRA, mit Zusätzen und Ergänzungen des Verfassers. Deutsch von Dr. ERNST LANELA. Sonderabdruck aus dem *Archiv der Mathematik und Physik*, III. Reihe, Band XXII, Heft 2/3. B. G. Teubner. Leipzig.

Leçons sur l'Intégration des Equations différentielles aux Dérivées partielles. Par VITO VOLTERRA. Professées à Stockholm. Nouveau tirage. Paris, Hermann. 1912. 3 + iv + 83 pp.

The first of these books consists of three lectures delivered at Clark University, and afterwards printed by the University.* They have since appeared in the second form in German in the *Archiv der Mathematik und Physik*, (3), 22 (1914), pages 97–182. In the latter form some of the details omitted in the original are supplied. The third book is a reprint of lectures delivered at Stockholm in 1906. There have been added some corrections, and some bibliographical notes. These lectures are striking examples of the intimate relationship between the advance of mathematics and that o physics.

The fundamental notion of the Stockholm lectures is that the theories of the propagation of heat, of hydrodynamics, elasticity, Newtonian forces, and electromagnetism can all be treated from a single point of view—reducing indeed to differential equations of the same general form but of three types, the facts and the processes used following the types. A good supplementary paper to read along with the first part of the lectures on differential equations, containing examples and more detail, is to be found in the *Annales de l'Ecole Normale*, (3), 24 (1907), page 411. The most interesting part of the lectures is the introduction of the notion, due to Professor Volterra, of *function of a line*. In the fifth lecture this notion appears, and is indeed the guide to a generalization of the

* The volume also contains lectures by Rutherford: "History of the alpha-rays from radio-active substances"; Wood: "The optical properties of metallic vapors"; Barus: "Physical properties of the iron carbides."

analytic functions of a complex variable. The particular function of a line used in this lecture is the line integral

$$V = u_0 + \int_A^B (Xdx + Ydy + Zdz),$$

where u_0 is a constant and the line integral extends from a point A to a point B. If we set $X' = (\partial Y/\partial z - \partial Z/\partial y)$, $Y' = (\partial Z/\partial x - \partial X/\partial z)$, $Z' = (\partial X/\partial y - \partial Y/\partial x)$, then, n being the outward normal to the surface enclosed, the line integral in question around a loop will, by Stokes's theorem, be the same as

$$W = \int\int (X' \cos nx + Y' \cos ny + Z' \cos nz)dA.$$

If we let the loop decrease and determine the limit of the ratio of W to the area enclosed, as the vanishing loop approaches a point, the limit in question is nothing else than the projection, on the normal to the surface at the point, of the vector whose components are X', Y', Z', that is, of the *curl* of the vector X, Y, Z. This projection, if found for a point on the path of integration, Professor Volterra calls the derivative of the function of the line V with respect to the surface, and he represents the curl by the symbols

$$X' = \partial V/\partial(yz) \quad Y' = \partial V/\partial(zx) \quad Z' = \partial V/\partial(xy).$$

If now there is a function u whose gradient is (X', Y', Z'), that is, if

$$\partial u/\partial x = X', \quad \partial u/\partial y = Y', \quad \partial u/\partial z = Z',$$

then the convergence of the gradient of u is zero, and u is harmonic since $\nabla^2 u = 0$. When these relations are satisfied we have u and V so related that one may be called the conjugate of the other. This is the generalization referred to of the theory of complex variables. An easy mathematical example is found by setting

$$u = x^2 + y^2 - 2z^2,$$

which is harmonic, and whose gradient is the vector

$$(2x, 2y, -4z).$$

It is conjugate to the function (integral around a loop)

$$V = \int (2zydx - 2zxdy),$$

since the curl of $(2zy, -2zx, 0)$ is $(2x, 2y, -4z)$, the gradient of u. An easy physical example is the field of potential at a fixed origin of a single magnet pole in empty space, as the pole is moved into all possible positions, which gives the function u; and the field of a circuit carrying a unit electric current, as it is moved into all possible positions and shapes, which gives the potential V at the fixed origin. The two functions are conjugate, the first a function of a point, the reciprocal of its distance from the origin; the latter a function of a line, the solid angle it subtends at the origin.

The function of a line of course need not be conjugate to a function u. In case X', Y', Z' is a vector whose convergence is zero, then it may be written as the curl of a vector X, Y, Z, the well-known relation of a vector to its vector potential; and the integral

$$\iint (X' \cos nx + Y' \cos ny + Z' \cos nz)dA = \int (Xdx + Ydy + Zdz)$$

gives a function of a line. It need be remarked that these definitions of function of a line and the differential of a function of a line are generalized somewhat in the calculus of fonctionnelles.

The notion of monogenicity is extended, under the name isogenicity, to space of three dimensions in the following form. Two functions of a complex variable z are monogenic if their differentials at any point have a ratio f which is a function of the point but not of the direction of the differentials. Two functions of a line are isogenic if at each point of the line the vector derivative of each, that is, the curl of the vector which is to be integrated along the line, is parallel to the vector derivative of the other. In vector notation we would write the definition

$$V \nabla B_1 = fV \nabla B_2,$$

where f is independent of the derivative plane as defined above. The idea of function of a line may evidently be extended to functions of surfaces and hyperspaces, and leads into the calcul fonctionnel.

These functions of lines and surfaces, and for the most general case, of hyperspaces, enable one to understand the application of the methods of Jacobi and Hamilton to the

problems of the calculus of variations. The simple integral of the ordinary theory may be considered as a function of its limits and of the values of unknown functions at the limits. The extensions to multiple integrals are easily suggested by this view of the procedure, namely, we must consider the integrals to be functions of the lines, or surfaces, or hyperspaces, that bound the space, and of the values of unknown functions on these contours.

The detailed treatment of the partial differential equations is to be found in the last four lectures, and reference to them is necessary to have a clear notion of them.

In the course at Clark University we find the dominant idea again to be the unifying principles of the application of mathematics to mechanics, elasticity, and mathematical physics. The first lecture is devoted to showing the reduction of physical problems to problems in the calculus of variations, the second to the advance in methods in elasticity, and the third to the problem of heredity, which leads to Volterra's integro-differential equations. These three lectures we will examine in some detail.

In the first there is given a reduction of the problem of electrodynamics into terms of the variation of the definite integral

$$P = \int dt \tfrac{1}{2} \int\int \Sigma(\alpha_{rs} X_r X_s + \beta_{rs} L_r L_s) dA,$$

in which the quantities α, β are quite arbitrary. This variation in terms of purely arbitrary quantities shows us that we can devise an infinity of mechanical explanations or models of electrodynamic phenomena. But this analytic form of the problem enables us to introduce curvilinear coordinates, and thus consider curvilinear spaces. We also are enabled to find the integral invariants, and to apply Volterra's reciprocity theorem which corresponds to Green's theorem, and further to introduce the generalization of the Hamilton-Jacobi methods and the principle corresponding to stationary action and varying action. In order to accomplish this, the functions of lines, surfaces, and hyperspaces have to be used. We thus come into contact with the theory of the inversion of definite integrals, that is, the solution of linear integral equations, and with the study of functions of variables which run over assemblages of curves, of surfaces, etc. We are brought up to the functional calculus, or as it has been called, general

analysis. The generalization of the Hamilton-Jacobi method replaces the canonical equations by partial derivatives, and the partial differential equation of Jacobi is replaced by a functional equation.

There is an intimate connection between the partial differential equation, its characteristics, and the theory of waves.* It is simply necessary to consider the variable t as on the same footing as the variables x, y, z. The bearing of this is shown in the closing considerations of the first lecture. For example, in the very simple case of the equation of a vibrating membrane

$$\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 = 0$$

we have the partial differential equation corresponding to the vanishing of the variation of the integral

$$V = \int\int\int [(\partial u/\partial t)^2 - (\partial u/\partial x)^2 - (\partial u/\partial y)^2] dx dy dt.$$

The surfaces of discontinuous derivatives and variation of V always equal to zero are then the envelopes of the characteristic cones of the partial differential equation. The general question of waves is therefore considered, and Minkowski's universe noticed. This naturally leads to the Lorentz transformation, and to Poincaré's demonstration that under this transformation the integral whose variation was considered, and which may be called the action, remains invariant.

In the second lecture the development of methods in the theory of elasticity is taken up, particularly those that are connected with the ideas already mentioned. The two great classes of methods of integrating the differential equations of elasticity may be called the method of Green with its extensions, and the method of simple solutions. Green's method further has two divisions, in one of which the conception of Green alone is sufficient to solve the problem, in the other we must add consideration of the characteristics. Green's method starts with Laplace's equation, and depends upon a reciprocity theorem, by means of which from a fundamental solution one is enabled to determine a harmonic function inside a given region when its values on the contour are given. Betti carried the method of Green over into elasticity and extended the reciprocity theorem by the proposition: If two

* Encyclopédie des Math., II, 1 (II, 22, 8).

systems of exterior forces determine two systems of displacements in an elastic body, the work done by either in producing the displacement due to the other is the same.

The second division of methods along the line started by Green is found in Kirchoff's work on the equation of retarded potential on four variables. He succeeded in solving it by means of a fundamental solution due to Euler, but in order to perceive the real difficulties in the way it is necessary to consider the problem on only three variables. The characteristics enter into the solution radically in this case. The lecturer shows the inherent difference between the cases of three dimensions and four dimensions. Again when the body is not simply connected and the functions can be polydromic the method of Green needs further extensions. The reciprocity theorem enters in as a theorem of the symmetry of a set of coefficients $E_{ij} = E_{ji}$, which occur in the linear equations that give the efforts in terms of the distortions.

The method of simple solutions has been given great power by the development of the theory of integral equations, and the determination of methods of expansion in series of orthogonal functions.

The third lecture introduces the new developments due to Professor Volterra himself and now well-known. These lead to the division of mechanics into the mechanics of no heredity, wherein the state of a system depends only upon the infinitesimally near states preceding, and the mechanics of heredity, in which the state depends upon all the preceding states, thus introducing an action at a time-distance. A simple example is used to illustrate the new problem, the dependence of the angle of torsion of a wire upon the moment of torsion. Instead of Hooke's law

$$\omega = KM,$$

we find it must be expressed by a more elaborate law dependent upon the time

$$\omega = KM(t) + \int_{t_0}^{t} M(t)\varphi(t, \tau)d\tau + \frac{1}{1\cdot 2}\int_{t_0}^{t} d\tau_1 \int_{t_0}^{t} d\tau_2 M(\tau_1)M(\tau_2)\varphi(t, \tau_1, \tau_2) + \cdots.$$

In the case of linear heredity this expression terminates with the second term. There arise now several questions.

(1) What is the significance of the coefficient $\varphi(t, \tau)$?
(2) When φ is known, how may M be found for a given ω?
(3) How is φ determined when it is unknown?
(4) Is it possible to extend these conceptions to the general problem of elasticity?
(5) Is it possible to extend these conceptions to the phenomena of magnetism and electricity?
(6) What phenomena will be amenable to this method of treatment?

For the first three questions it is found that the methods of integral equations are sufficient to furnish the answer. However for the fourth a new type of equation is in evidence, the integro-differential equation. We meet this indeed in the problem of the wire itself when we study its oscillations. The integro-differential equation involves the partial derivatives of the unknown function as well as integrals containing the unknown function. The answer to the fifth question leads also to integro-differential equations for the electromagnetic equations.

To resolve equations of this form a further extension of the method of Green is necessary, and a new reciprocity theorem arises. Fundamental solutions may be found and from these arises the complete solution.

In applying algebra to natural phenomena we have the great advantage that we can postpone to the last moment the specification of the constants that enter the natural problem, and are thus in a much better position to consider the related hypotheses. So too in the functional calculus in all its forms, we are able to postpone the specification of the functions entering the problem until the last moment, and are thus able to avoid vicious hypotheses in the beginning. Without these developments many types of problem become impossible of solution. Further, advances in mathematical physics come from the subsumption under one law of many different classes of phenomena.

In the translation of the Clark University Lectures many details of the transformations in the first lecture are supplied, to the relief of the general reader. Further, the references have been made more precise and have been inserted on the pages as footnotes instead of being collected at the ends of the lectures. Some minor corrections have been made. The section headings have been inserted over the sections.

This edition of these very valuable lectures will be welcomed.

The whole field of functional calculus is a new territory but recently open for settlement, though an adventurous investigator occasionally explored small parts of it in the past century. The important extensions of mathematics have come from the problems of inversion, such as the Galois theory, theory of ideals, differential equations, integral equations, and now the calcul fonctionnel. These developments of Professor Volterra are of the highest importance mathematically aside from all of their physical interest, for the reason that they furnish a very practical path of entry into this new field and occupy a considerable part of the field itself. Fortunately we can follow them more in detail in the two recent courses of his lectures, Leçons sur les fonctions des lignes, and Leçons sur les équations intégrales.

JAMES BYRNIE SHAW.

SHORTER NOTICES.

Mysticism in Mathematics. By HASTINGS BERKELEY. Henry Frowde, Oxford University Press, 1910. vi + 264 pp.

IT may be well to begin by stating what Mr. Berkeley's thesis is not. The kind of mysticism which he thinks he detects in mathematics is not any of the kinds of mysticism that one encounters in the history and philosophy of religion. It is not contended that devotion to mathematics begets or tends to beget in the devotee a sense of an immediate and ineffable union or identification with deity. It is not argued that there is any essential likeness between Euclid and Timæus or between Gauss and Angelus Silesius. What Mr. Berkeley calls mysticism in mathematics is, he says, so called by him because no other name seems to him so appropriate. It appears to be impossible to state with mathematical precision what the thing is. Of course the author is not to be blamed for that. He succeeds in the difficult enterprise of making the matter as clear to the reader as it is to the writer, and that is all that can be reasonably expected.

As nearly as we have been able to make it out, the thesis may be broadly stated to be that, owing to a kind of reaction of language and especially of highly symbolic language upon

thought and belief, mathematicians tend to acquire a state of mind in which many purely fictitious things appear to be actual or real things. The thesis, which is very elaborately discussed, is approached from several points of view. The book falls into three parts.

In Part I the method is a priori. The Pythagoreans raised mathematics to the level of a science and the Pythagoreans were "enthusiasts a d mystics." The conjunction is suspicious. This and kindred considerations lead Mr. Berkeley to devote about 50 large pages to a very interesting psychological and linguistic dissertation designed to show that we are in constant danger of "the vicious reaction of words upon the process of thought," a reaction having a kind of hallucinatory effect, producing a "tendency to mysticism," a tendency to fancy that all symbols and modes of expression represent actualities instead of representing, as frequently they do, merely conventional or fictitious things.

The argument in Part I is general; it applies to all language. But the more symbolic the language, the more vicious is "the vicious reaction." And Part II (94 pages) is devoted to showing, on the one hand, that mathematicians by constantly talking about imaginaries and infinites in algebra and geometry lose sight of the fictitious and convential character of such things, and, on the other hand, to showing how algebra and geometry may be thoroughly purged of mystical elements.

Part III (113 pages), which deals with metageometry, arrives at the interesting conclusion that it "is an illusion" to suppose "that we can conceive different kinds of distance" or different kinds of space, as euclidean and non-euclidean. The illusion is "engendered by an unrealized reaction of symbolic forms, verbal or algebraic, on the thinker's judgment of his own process of thought."

Stylistically the book is well done. Its message to mathematicians—we mean its caution about treating such fictions as the circular points, for example, as real entities in space—might have been more effectively delivered in a score of pages. If it were shorter, the work would be more useful even for philosophers and other laymen.

CASSIUS J. KEYSER.

Exercices de Géométrie analytique. Par P. AUBERT and G. PAPELIER. Tome Premier. Paris, Libraire Vuibert. 360 pp.

IN this book are given 451 exercises on analytic geometry divided among the following fields: the straight line, 94 exercises; the circle, 69 exercises; parametric equations, 46 exercises; classification of conics, 55 exercises; the general equation of the conic, 31 exercises; centers and diameters of conics, 34 exercises; the general properties of conics, 91 exercises; polar coordinates, 31 exercises. Many of the exercises are accompanied by a figure and about one half of them are solved. In several cases more than one solution is given, so as to illustrate different methods of approaching the same problem. The authors have assumed the student to be familiar with harmonic properties, projective transformations, and similar topics which are given in the United States in a second course in analytics. Homogeneous coordinates are not used, and thus many of the problems are solved by rather tedious methods. In our American colleges the book could be used to good advantage in a second course in analytics in conjunction with such a book as Salmon's Conic Sections. The problems are of too difficult a nature for a first course as given in our institutions.

F. M. MORGAN.

Orders of Infinity. The "Infinitärcalcül" of Paul Du Bois-Reymond. By G. H. HARDY. Cambridge University Press, 1910. viii + 62 pp.

MANY problems of mathematical analysis are adequately treated by consideration of the fact that a function does or does not approach a limit, where we may understand the notion of a limit to include the real infinity, with or without sign, and the complex infinity. Other questions, however, demand more refined investigations,—on the one hand when a limit exists and an estimate is necessary of the rapidity of approach to the limit; on the other hand when no limit exists and some notion of the behavior of the function is still requisite. As early as 1821, Cauchy recognized the usefulness of such considerations in his accurate definition of la plus grande des limites and la plus petite des limites,—today more commonly known as superior and inferior limits.

A great many theorems dealing with the limit-behavior of functions were obtained by Du Bois-Reymond in a series of papers dating from 1871 to 1880. These theorems have been collected, recast according to modern requirements of rigor, and amplified by Hardy in No. 12 of the Cambridge Tracts in Mathematics and Mathematical Physics. The work has been done in a manner admitting of no criticism; the treatment is clear and readable; the proofs are accurate and carefully worded.

Questions of the sort considered are generally reducible to a comparison of the rates of increase of positive real functions, as the positive real independent variable becomes infinite (continuously or over integral values); into this form most of the results are thrown by the author. A considerable part of the tract deals with logarithmico-exponential functions,—those obtained by rational operations, the extraction of roots, and the taking of logarithms and exponentials. Such functions have various properties which render their study easy and important; for instance, they always admit of comparison as regards rate of increase, and relations of comparison may under simple conditions be differentiated and integrated.

The tract contains a sketch of applications to such questions as the convergence of series and integrals, asymptotic formulas, the distribution of prime numbers, and the theory of integral functions of a complex variable.

One is impelled to wonder how much of the fairly extensive notation introduced will be found really desirable in actual use of the results. The notions of inferior and superior limit have won a permanent place; Landau's symbols $O(f)$ and $o(f)$ have in a short time come into such wide use as apparently to insure their retention. It is doubtful whether any further notation will be found necessary.

The tract is printed with the clearness characteristic of the series. The only typographical inaccuracy noted is on page 42, line 12, where $e_{\Delta x}$ should be replaced by $e^{\Delta x}$.

WALLIE ABRAHAM HURWITZ.

The Hindu-Arabic Numerals. By DAVID EUGENE SMITH and LOUIS CHARLES KARPINSKI. Boston, Ginn and Company, 1911. vi + 160 pp.

THE origin and development of our present number system is probably given as little thought as anything that is so

commonly known and used. For persons who are not interested in things mathematical, as well as for those who are thus interested, The Hindu-Arabic Numerals furnishes information that is by no means current. The authors have given in concise form the history of the characters that every school boy of modern days learns to use in computation. Beginning with an account of the early ideas about these characters, the writers go on with the details of place value, the symbol zero, and the actual physical forms by which the numbers have been represented at various times. In chapter 5 is begun the account of their introduction into Europe. This westward aggression seems to have been very slow at first because the new numerals did not appeal to the traders and were regarded as a sort of novelty by educators. The complete acceptance of this really wonderful system finally took place in the sixteenth and seventeenth centuries.

J. V. McKelvey.

Vorlesungen über darstellende Geometrie, Band I. Dr. F. v. Dalwigk. Leipzig, Teubner, 1911. xvi + 360 pp.

Descriptive geometry is presented in this book in a thoroughly instructive manner. A noticeable balance is preserved between what might be called theoretical and practical. Methods of treatment characteristic of pure geometry are used freely, but at the same time the technical use of the principles involved is given due attention. The frequent comparison of different methods of making a given construction is worthy of mention. Shadow construction is treated with excellent simplicity and completeness throughout the book. The treatment of the projection of the intersection of two surfaces is such as to give the beginner a good introduction to the theory of space curves.

In the latter part of the text, trihedral angles with applications to astronomy, surfaces of revolution, screw surfaces, stereographic projection, parallel perspective, and rectangular coordinates are taken up in some detail. The last section of the book is an introduction to mechanical drawing.

J. V. McKelvey.

Darstellende Geometrie. By TH. SCHMID, associate professor of geometry at the technical school of Vienna. (Sammlung Schubert, LXV.) Berlin and Leipzig, G. J. Göschen, 1912. vi + 279 pp., and 107 figures.

LIKE the other volumes in the Schubert series, this book is written for beginners, and does not attempt to be an exhaustive treatise. It begins with the ordinary double orthogonal projection, but proceeds rather rapidly, so that only forty pages are required for the usual presentation of rectilinear figures. Considerable emphasis is laid on proportion, and the criteria for determining which of two perspective objects hides the other. A generous number of exercises is added at frequent intervals. At the end of this first part is a good introduction to the theory of shades and shadows, and its application to polyhedra. The second and third chapters treat of the sphere, cylinder, and cone; they comprise two thirds of the volume. The discussion begins with polarity as applied to the circle. Much of the text is divided into two columns, one describing polarity, and the other antipolarity, the product of polarity and a reflection. The first twenty pages are very elementary, but the subject is so presented that problems of considerable complexity are disposed of as easily as the simpler ones. An elementary knowledge of plane analytic geometry is presupposed, but most of the properties of conics that are wanted later are established. In connection with the discussion of curvature use is made of the derivative, and in the rectification of the ellipse a foot-note is provided which expresses the length of arc as an elliptic integral. Mechanical devices for drawing conics are described, and the theory of each explained. A section on the illumination of the sphere is delightful reading. Not only does it present the theory in a clear and concise way, but enlivens it with a discussion of the reasons for including the theory in the discussion; both practical and esthetic grounds are presented that are quite convincing. Different degrees of illumination are expressed quantitatively, and directions given for preparing washes to procure them.

An unusual feature is the extent to which the developments of various space curves on the cylinder and cone are discussed. We meet the conchoid, cardioid, cissoid, serpentine, sine curve, and a number of less well known plane curves as developed from an algebraic intersection.

The detailed discussion of the space quartic curve is also somewhat of a departure, but one that is justified. Without using any other machinery than that provided in the previous part of the text, the appearance of the elliptic, nodal, and cuspidal quartics is found, and some of the properties derived. Here again the theory of shades and shadows is liberally applied.

The last chapter, comprising about one sixth of the volume, is devoted to orthogonal axonometry. Here the presentation is not so successful. Too much detail is given, and the reader does not see readily the advantage of the whole procedure. The elementary projections are given as before, in fact several figures are almost identical with those found in the earlier presentation. After twenty pages, however, the treatment is much more satisfactory. The purpose now appears, and the whole discussion is easier and clearer. A detailed treatment of the determination of the scale in each of three directions is followed by the representation of polyhedra, then by complicated architectural figures. A short discussion of the three round bodies is added, and the results applied to an extension coupling. Only a small number of exercises are given in this last part.

VIRGIL SNYDER.

Die graphische Darstellung. By FELIX AUERBACH. Leipzig and Berlin, B. G. Teubner, 1914. (Aus Natur und Geisteswelt, volume 437.) 97 pp. and 100 figures.

THIS little volume is written for readers without mathematical training. Its purpose is to compare the value of various methods of representation. It compares magnitudes by means of lengths of segments of lines, areas of rectangles with constant bases, areas of squares, circles, sectors, and even volumes of certain solids. Incidentally a mass of statistical information is given in the illustrations. After these preliminary notions follows a full and readable discussion of the idea of a plane curve, including periodic and general trigonometric representations. Finally, automatic tide registers, magnetic fields, seismographs, registering thermometers, solar photographs, and sound waves are briefly discussed.

VIRGIL SNYDER.

Differential- und Integralrechnung. Erster Band: Differentialrechnung. By FR. W. MEYER. Second edition. Berlin and Leipzig, G. J. Göschen, 1912 (Sammlung Schubert, X). xv + 418 pp.

THE first edition of this book was published in 1901. In the present edition a large number of details are altered, but the general development is the same as in the earlier one. Thus, old Art. 12 is now pushed into the first chapter and made Art. 6. The new Art. 15 is much shortened. Maxima and minima of functions of more than one variable are introduced. An appendix is provided which treats of tangents and normals.

VIRGIL SNYDER.

NOTE ON "THE DISCOVERY OF INVERSION."

IN a review* under this title of Professor Bützberger's monograph, Ueber bizentrische Polygone, etc., I referred to a theorem on generalized Steinerian series of circles, which I proved in 1901 in connection with an application of elliptic functions to certain closed linkages.

Since the appearance of this review my attention has been drawn to the fact that A. Hurwitz had previously stated the main part of the theorem in an article, Ueber die Anwendung der elliptischen Funktionen auf Probleme der Geometrie, which appeared in the *Mathematische Annalen*, volume 19, page 65.

As will be noticed, Hurwitz, without proof, merely states the theorem, and his method by which it can be proved is entirely different from that of closed linkages.

This note also applies to a similar statement concerning the same theorem in the "Sprechsaal† für die Encyklopädie der mathematischen Wissenschaften."

ARNOLD EMCH.

CORRECTION.

PROFESSOR Veblen has kindly called my attention to an error in my review of "Monographs on Topics of Modern Mathematics" in the BULLETIN for January, 1914. The

* BULLETIN, vol. 20, p. 413.

† *Jahresbericht d. Deutschen Mathematiker-Vereinigung*, vol. 23, 2. Abt., Heft 3/4, pp. 50–51.

criticism against the proof of theorem 19 in the first paper is invalid; the proof as it stands is accurate and complete. Professor Veblen believes also that it is legitimate to use the symbol $\{ABC\}$ in the two ways employed, the language in the one case (as in theorem 7) being essentially an abbreviation of that in the other case (as in assumption II).

R. D. CARMICHAEL.

NOTES.

AT the November meeting of the Edinburgh Mathematical Society the following papers were presented: By G. N. WATSON: "The convergence of the series in Mathieu's functions"; by D. M. Y. SOMMERVILLE: "Taylor's cubics associated with a triangle in non-euclidean geometry"; by E. L. INCE: "The elliptic cylinder functions of the second kind."

THE annual meeting of the London mathematical society was held November 17, and the following officers elected for the year: president, Sir J. LARMOR; secretaries, J. H. GRACE, T. J. I'A BROMWICH; treasurer, A. E. WESTERN. Professor A. E. H. LOVE delivered an address on "Research in mathematics." Arrangements were made to hold regular monthly meetings during the year. The society now numbers more than three hundred members.

THE summer meeting of the Deutsche Mathematiker-Vereinigung, which was to have been held in September in affiliation with the Hanover meeting of the German naturalists and physicians, was omitted. No provision has been announced for the next meeting.

CONTRARY to the announcements made in the newspapers, nearly all the European universities and technical schools are open as usual. American students provided with passports are welcome.

THE sixty-sixth meeting of the American association for the advancement of science was held in Philadelphia during the week beginning December 28, under the presidency of Dr. C. W. ELIOT.

At the meeting of Section A, the retiring vice-president, Dr. FRANK SCHLESINGER, presented his address on "The object of astronomical and mathematical research." Professor H. S. WHITE is vice-president, and Professor F. R. MOULTON secretary of Section A.

A REGULAR meeting of the National academy of sciences was held at Chicago, December 7, 8, 9; the following papers in mathematics were presented at this meeting: By G. A. BLISS, "A generalization of a theorem of Gauss concerning geodesic triangles"; by L. E. DICKSON, "Recent progress in the theories of modular and formal invariants"; by G. A. MILLER, "The ϕ-subgroup of a group of finite order"; by E. H. MOORE, "On the integration by successive approximations of the ordinary differential equation of the first order in general analysis"; by F. R. MOULTON, "An extension of the process of successive approximation for the solution of differential equations"; by H. S. WHITE, "The synthesis of triad systems Δ_t in t elements, in particular for $t = 31$"; by E. J. WILCZYNSKI, "Conjugate systems of space curves with equal Laplace-Darboux invariants."

THE royal gold medal of the Royal Society of London has been awarded to Professor E. W. BROWN, of Yale University, in recognition of his investigations in mathematical astronomy.

THE technical school of Zürich has conferred its honorary doctorate on Professor H. A. SCHWARZ, of the University of Berlin, on the fiftieth anniversary of his receiving the doctor's degree.

PROFESSOR P. KOEBE, of the University of Leipzig, has accepted a professorship of mathematics at the University of Jena, as successor to Professor J. THOMAE.

DR. W. DEIMEL, of the technical school in Munich, was killed in battle October 23, at the age of 28 years.

AT the University of Illinois, Mr. N. C. GAINES has been appointed instructor in mathematics; Mr. W. H. WILSON and Mr. R. H. MARSHALL have been appointed assistants in mathematics.

On his seventieth birthday, August 25, 1914, the Teachers' association of South Africa presented to Dr. Thomas Muir an illuminated address and a gift of books.

Professor Frederick Anderegg, of Oberlin College, has returned from a year's leave of absence.

Mr. W. E. Edington, of the University of Colorado, has been appointed professor of mathematics in the University of New Mexico.

At Pennsylvania State College, Dr. E. R. Smith has been promoted from an assistant professorship to an associate professorship of mathematics.

At the University of Texas, Dr. D. F. Barrow has been appointed instructor in applied mathematics, and Mr. F. A. LaMotte instructor in pure mathematics.

Professor R. G. D. Richardson, of Brown University, has been elected a member of the section of mathematics and astronomy of the American academy of arts and sciences.

At Oberlin College, Miss M. E. Wells has been appointed acting associate professor of mathematics.

Professor G. B. Guccia, of the University of Palermo, founder of the *Rendiconti del Circolo matematico di Palermo*, died October 29, 1914, at the age of 59 years.

Professor R. Lehmann-Filhés, of the University of Berlin, died June 5, at the age of 60 years.

The death is reported of Professor Heinrich Burkhardt, of the technical school of Munich, in his fifty-fourth year.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

APOLLONIUS OF PERGA. Treatise on conic sections; edited in modern notation with introduction including an essay on the earlier history of the subject by Sir T. L. Heath. Oxford, Clarendon Press, 1913. 8vo. 424 pp. Cloth. 20s.

BOHANNAN (R. D.). Plane analytical geometry. Columbus, O., R. G. Adams, 1913. $1.60

CARRARA (B.). La s. scrittura i ss. padri e Galileo sopra il moto della terra, ossia il sistema copernicano e la s. scrittura al tempo di Galileo: monografia. Seconda edzione, migliorata ed accresciuta. Milano, G. Palma, di G. Daverio, 1914. 16mo. 170 pp. L. 1.50

CAUER (D.). Neue Anwendungen der Pfeifferschen Methode zur Abschätzung zahlentheoretischer Funktionen. (Diss.) Göttingen, 1914.

COBB (C. W.). Plane analytic geometry. Norwood, Mass., Norwood Press, 1913. $1.00

ECHOLS (W. H.). Evolutionary constructions of the imaginary power of a number and its expression as an exponential function. Charlottesville, 1913.

EMMERICH (A.). Kubische Aufgaben aus der Stereometrie. Nebst Anleitungen zur Lösung, Determinationen und den Ergebnissen der Zahlenbeispiele. Weinheim, 1914. 8vo. 8 + 127 pp. M. 2.50

FABARIUS (W.). L. Euler und das Problem Fermats. Cassel, 1914. 8vo. 12 pp. M. 1.20

GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie. II. Teil: Die analytische Geometrie des Raumes. 5. Auflage. Leipzig, Teubner, 1913. 8vo. Geb. M. 3.00

GROOT (W. F. DE). De theorie der kwadratische vormen van oneindig vele veranderlijken en haar toepassing op de lineaire integraalvergelijkingen. Leiden, 1913. 8vo. 106 pp.

HEATH (T. L.). See APOLLONIUS.

HEILAND (F.). Eine Gruppe von Kurven 3ter Ordnung. Progr. grossherzogl. Reformrealgymnasium, Apolda, 1914.

KREUSER (P.). Ueber das Verhalten der Integrale homogener linearer Differentialgleichungen im Unendlichen. (Diss.) Tübingen, 1914.

LABOUREUR (M.). Cours de calcul algébrique différentiel et intégral. Paris, 1913. Fr. 15.00

LESSER (O.). Die Infinitesimalrechnung im Unterricht der Prima. Zweite verbesserte Auflage. Berlin, Salle, 1912. Geb. M. 2.20

LINDWART (E.). Ueber eine Methode von Laguerre zur Bestimmung des Geschlechts einer ganzen Funktion. (Diss.) Göttingen, 1914.

LOUP (W. M.). Een bijdrage tot de theorie der cyclische oppervlakken en congruenties. (Thesis.) Utrecht, van Boekhoven, 1914. 4to. 80 pp.

MERZ (K.). Parallelflächen und Zentralfläche eines besonderen Ellipsoides und die Steinersche Fläche. Beispiele einer quadratischen Transformation. Chur, 1914. Gr. 8vo. 54 pp.

MOZANS (H. J.). Woman in science. New York and London, Appleton, 1914. 8vo. 13+452 pp. $2.50

PAULHAN (F.). Esprits logiques et esprits faux. Seconde édition. Paris, Alcan, 1914. 8vo. 8+388 pp.

PEIRCE (B. O.). A short table of integrals. Abridged edition. Boston, Ginn, 1914. 8vo. 32 pp. Paper. $0.15

PIERPONT (J.). Functions of a complex variable. Boston, Ginn, 1914. 8vo. 14+583 pp. Cloth. $5.00

POND (R. S.). Collineations in space of four dimensions. Lawrence, Kan., University of Kansas, 1913.

RITTER (H.). Beweis des Fermatschen Satzes. Cassel, 1914. 8vo. 12 pp. M. 0.50

ROBERT (E.). Composition des formes quadratiques de quatre et de huit variables indépendantes. Zürich, 1912. 8vo. 56 pp.

RUDIO (F.). See GANTER (H.).

SCHMEHL (C.). Die Algebra und algebraische Analysis nebst einer elementaren Theorie der Determinanten und den Anfangsgründen der Differentialrechnung in den oberen Klassen von höheren Lehranstalten. 2te Auflage. Giessen, E. Roth, 1914. 316 pp.

SPORER (B.). Ueber geradlinige Transversalen algebraischer Kurven. Wien (*Sitz. Akad.*), 1914. Gr. 8vo. 34 pp.

WEBER (O.). Binormalenflächen mit einer zur Striktionslinie äquidistanten Asymptotenlinie in Beziehung zu Evoluten Ribaucourscher Kurven. (Diss.) Halle, 1914.

WILENSKY (M.). Ueber Besselsche Funktionen. (Diss.) Bern, 1914.

ZEUTHEN (H. G.). Lehrbuch der abzählenden Methoden der Geometrie. Leipzig, Teubner, 1914. 8vo. 12+394 pp. Geb. M. 17.00

II. ELEMENTARY MATHEMATICS.

BAKER (W. M.) and BOURNE (A. A.). Examples in arithmetic extracted from the Public School Arithmetic. London, Bell, 1914. 8vo. 12+217+52 pp. Cloth. 2s.

BELL (A. H.). A course of geometry theoretical and practical. London, Rivingtons, 1914. 8vo. 6+127 pp. 2s. 6d.

BETZ (W.) and WEBB (H. E.). Plane geometry. Boston, Ginn, 1912. 12mo. 10+332 pp. Cloth. $1.00

BOURNE (A. A.). See BAKER (W. M.).

BUKER (E. F.) and FELTER (W. L.). Arithmetics. 3 books. Books 1–2. New York, Silver, Burdett & Co., 1913. $0.80

CAJORI (F.). School arithmetics; primary book. New York, Macmillan, 1914. $0.35

CAMMAU (P.) et HUOT (J.). Algèbre et géométrie. Classe de première et de seconde A et B. 2me édition. Paris, 1913. Fr. 2.60

CHIGNELL (N. J.). Numerical trigonometry. Oxford, Clarendon Press, 1914. Cr. 8vo. 140 pp. 2s. 6d.

COATES (J. V. H.). A first book of geometry. London, Macmillan, 1912. 8vo. 12+142 pp. 1s. 6d.

COX (H. C.), TALLANT (G. A.) and UMHOLTZ (F. H.). Modern arithmetic, elementary. Chicago, Lyons & Carnahan, 1913. 6+443 pp. $0.60

DRONKE (J.) und LÖTZBEYER (P.). Mathematische Aufgaben für die Oberstufe höherer Lehranstalten. Dresden, Ehlemann, 1914. 230 pp.

DURELL (F.). Algebra. Book 1. New York, Merrill, 1914. $1.00

FAZZARI (G.). Elementi di aritmetica con note storiche e numerose questioni varie per le scuole medie superiori. Parte prima: Numeri interi. Operazioni, divisibilità, numeri primi. 2a edizione. Palermo, Reber, 1911. 12mo. 8+127 pp. L. 1.40

FELTER (W. L.). See BUKER (E. F.).

FERGUSSON (J. C.). Fergusson's percentage trigonometry or plane trigonometry reduced to simple arithmetic, with a short description of his percentage compass. London, Longmans, 1914. 8vo. 155 pp. Cloth. 3s. 6d.

FITE (W. B.). Second course in algebra. Boston, Heath, 1914. $0.90

GALDÉANO (Z. G. DE). Congreso de la enseñanza matematica celebrado en Paris del 1 al 4 de Abril. Conferencia dada en la Facultad de Ciencias de Zaragoza el dia 22 Abril. Zaragoza, 1914. 8vo. 39 pp.

GERLACH (A.). Lebensvoller Rechenunterricht. 1ter Teil: Einführung und Unterstufe. Leipzig, Voigtländer, 1914. 215 pp. M. 4.25

GIFFORD (MRS. E.). Natural sines to every second of arc and eight places of decimals. Published by Mrs. Gifford, Oaklands, Chard, Somerset, 1914. 6+543 pp. 20s.

GILLESPIE (F. C.). Worked exercises in elementary geometry. London, Milford, 1914. 3s. 6d.

GRABOWSKI (J.). Benedykta Herberta Arithmetica linealis, Cracoviae, 1577. Krakowie, 1913. 15 pp.

HANSEN (H. J. P.) og RASMUSSEN (H.). Geometriske øvelser. 2 Teile. Kopenhagen, Gyldendalske Boghandel, 1914. 32+70 pp.

HARVEY (L. D.). Essentials of arithmetic. Books 1–2. New York, American Book Co., 1914. $0.25+$0.50

HERRICK (F. M.). Arithmetical and geometrical calculations by analytical statement and comparative cancellations. Princeton, Ill., Herrick & Horton, 1914. 143 pp. $1.00

HUOT (J.). See CAMMAU (P.).

JACOB (J.). Praktische Methodik des mathematischen Unterrichts. Mit einem Vorwort von E. Mach. Wien, 1914. M. 3.40

JACQUET (E.) et LACLEF (A.). Arithmétique du brevet élémentaire et des cours complémentaires. 12me édition. Paris, 1913.

JAHNKE (E.). Die Mathematik an Hochschulen für besondere Fachgebiete. (Abhandlungen über den mathematischen Unterricht in Deutschland, Band IV, Heft 7.) Leipzig, Teubner, 1911. 8vo. 6 + 55 pp. M. 1.80

KNAB (P.). Rechenbuch für höhere und mittlere Lehranstalten. 2te u. 3te Auflage. Freiburg i. Br., Herder, 1914. 240 pp. M. 3.00

KUNDT (F.). Arithmetische Aufgaben mit einem Anhange von Aufgaben aus der Stereometrie für Lyzeen (höhere Mädchenschulen) und die unteren Klassen der Studienanstalten. Leipzig, Teubner, 1914. Gr. 8vo. M. 2.00

LACLEF (A.). See JACQUET (E.).

LEIGH (C. W.). See PALMER (C. I.).

LEMOINE (A.). Arithmétique. Théorie, exercices et problèmes. 3e édition. Paris, 1913. Fr. 1.00

——. 160 leçons d'arithmétique. Théorie. 2800 exercices et problèmes. 2me édition. Paris, 1913. Fr. 4.00

LÖTZBEYER (P.). See DRONKE (J.).

MAIR (D. B.). Exercises in mathematics. With answers and hints. London, Macmillan, 1914. Cr. 8vo. 4s. 6d. Without answers and hints. 3s. 6d.

MAXSON (W. S.). Practical self-keyed fundamental number work. Boston, Hammett, 1913. 48 pp. Paper. $0.05

MINET (A.) et PATIN (L.). Cours pratique d'arithmétique, de système métrique et de géométrie. Edition rev. et augm. Paris, 1913.

NUNN (T. P.). The teaching of algebra including trigonometry. London, Longmans, 1914. Cr. 8vo. 7s. 6d.

PALMER (C. I.) and LEIGH (C. W.). Plane trigonometry with 5-place tables. New York, McGraw-Hill, 1914. 8vo. 156+132 pp. Cloth. $1.50

PATIN (L.). See MINET (A.).

PRICE (E. A.). Solutions of the exercises in Godfrey and Siddon's Shorter geometry. Cambridge, University Press, 1914. 8vo. Cloth. 4s. 6d.

RASMUSSEN (H.). See HANSEN (H. J. P.).

SCHLÖMILCH (O.). Fünfstellige logarithmische und trigonometrische Tafeln. 24te Auflage. Braunschweig, Vieweg, 1914. Cloth. M. 1.30

SCHRÖN (L.). Siebenstellige gemeine Logarithmen der Zahlen von 1 bis 108,000 und der Sinus, Cosinus . . . von 10 zu 10 Sekunden, nebst einer Interpolationstafel zur Berechnung der Proportionaltafel. 28te revidierte stereotyp-Ausgabe. Braunschweig, Vieweg, 1914. Half calf. M. 7.30

SLICHTER (C. S.). Elementary mathematical analysis. A textbook for first year college students. New York, McGraw-Hill, 1914. 8vo. 490 pp. $2.50

SMITH (T. C.). Problem papers in arithmetic for preparatory schools. London, Bell, 1914. Cr. 8vo. 1s. 6d.

SMOLEY (C. K.). Tables. 8th edition. New York, McGraw-Hill, 1914. 21+330 pp. $3.50

SPRAGUE (C. E.). Accountancy of investment, with which are incorporated logarithms to 12 places and their use in interest calculations. New York, Ronald, 1914. Half leather. $5.00

TALLANT (G. A.). See COX (H. C.).

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